Laminar Boundary Layers

SOE3211/2 Fluid Mechanics lecture 4

Boundary layer (A)

Boundary conditions for flow at a wall

$$\underline{u}_{rel} = \underline{0}$$

– the flow shares the velocity of the wall.

For a stationary wall, $\underline{u} = 0$

Thus, no matter what the flow is doing anywhere else, there must be a laminar region somewhere close to the wall.

The near wall region where the flow adapts to $\underline{u}_{rel} = \underline{0}$ is called the $\boxed{\text{Boundary Layer}}$

(Almost) everything of importance happens in the boundary layer – important effects on

- lift
- drag
- heat transfer

$\therefore$ going to spend quite some time on it.

In particular, want to determine *wall shear stress*.

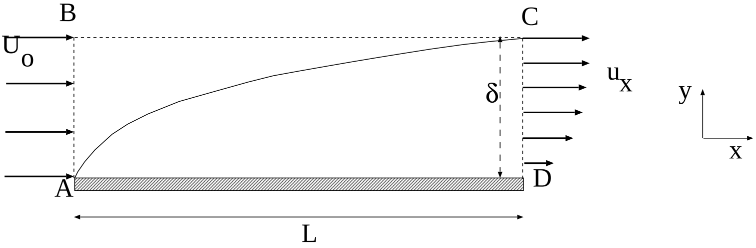
$$\tau_0 = \frac{F_0}{A}$$

Relate to friction coefficient

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U^2}$$

Laminar Boundary Layer (A)

Laminar flow around a flat plate. Fluid flows past the plate with velocity U_∞



Effect of the plate propagates outwards

$\Rightarrow$ broadening 'region of influence' around the plate.

We can define a distance δ – the *boundary layer thickness* – which is the distance at which the flow velocity begins to drop

Define δ as the distance that

$$u_x(\delta) = 99\% U_\infty$$

(the factor 99% is somewhat arbitary).

Flow is largely parallel to the plate except in 'adjustment region' around δ

Other definitions of δ possible

Skin Friction Coefficient C_f (A)

Define coefficient

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}$$

where τ_0 is the surface stress (force per unit area of the surface).

Useful : if we have C_f, can find force on the plate.

Blasius solution to NSE for laminar b.l. flow across plate gives τ_0, so

$$C_f = \frac{0.664}{\sqrt{\mathcal{R}e_x}}$$

with

$$\mathcal{R}e_x = \frac{U_0 x}{\nu}$$

and x is distance along plate. This is a *local* coefficient.

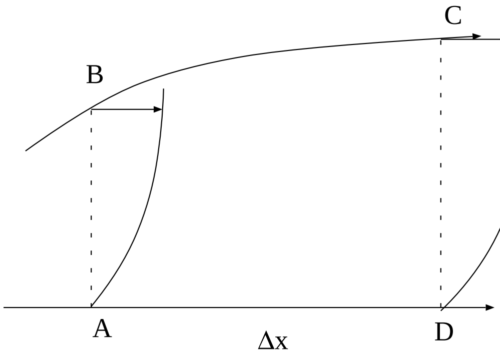
We want to be able to find the drag on a complete plate. Integrate along x, plate length L we get

$$\overline{C_f} = \frac{1.33}{\sqrt{\mathcal{R}e_L}} \quad \text{with} \quad \mathcal{R}e_L = \frac{U_0 L}{\nu}$$

Also : thickness of b.l.

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathcal{R}e_x}}$$

von Karman analysis (B)



Apply momentum equation $\rightarrow$

$$-\tau_0 \Delta x = \int_0^{\delta_2} \rho u_2^2 dy - \int_0^{\delta_1} \rho u_1^2 dy - \rho U_0^2 (\delta_2 - \delta_1)$$

In order :

$-\tau_0 \Delta x$	shear force opposing motion
$\int_0^{\delta_2} \rho u_2^2 dy$	rate of momentum transfer through CD
$-\int_0^{\delta_1} \rho u_1^2 dy$	same through AB
$-\rho U_0^2(\delta_2 - \delta_1)$	momentum through BC

But

$$U_0(\delta_2 - \delta_1) = \int_0^{\delta_2} u_2 \, dy - \int_0^{\delta_1} u_1 \, dy$$

Rearanging we find

$$-\tau_0 \Delta x = \rho U_0^2 \left[\int_0^{\delta_2} \left(\left(\frac{u_2}{U_0} \right)^2 - \frac{u_2}{U_0} \right) dy - \int_0^{\delta_1} \left(\left(\frac{u_1}{U_0} \right)^2 - \frac{u_1}{U_0} \right) dy \right]$$

Limit as $\Delta x \to 0$:

$$\tau_0 = \rho U_0^2 \frac{d}{dx} \int_0^{\delta} \left(\left(\frac{u}{U_0} \right)^2 - \frac{u}{U_0} \right) dy$$

or

$$\tau_0 = \rho U_0^2 \frac{d\theta}{dx}$$

where

$$\theta = \int_0^{\infty} \left(1 - \frac{u_x}{U_0} \right) \frac{u_x}{U_0} dy$$

Importance of this? If we know θ or u (measurements, theory), can calculate forces.

The quantity

$$\theta = \int_0^\infty \left(1 - \frac{u_x}{U_0}\right) \frac{u_x}{U_0} dy$$

is called the momentum thickness of the b.l. We can also evaluate

$$\delta^* = \int_0^\infty \left(1 - \frac{u_x}{U_0}\right) \, dy$$

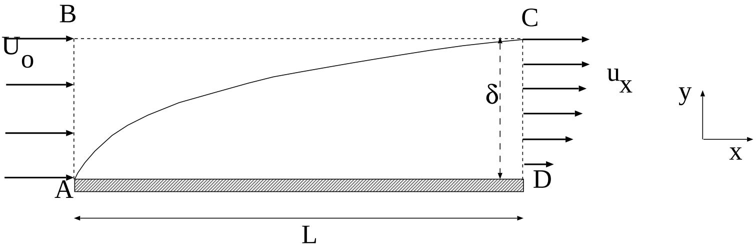
the displacement thickness.

The displacement thickness is the thickness that the b.l. fluid would occupy if the wall was not there, for the same mass flux.

The momentum thickness is the same but for the momentum flux.

Blasius Solution (B)

Mathematical solution for laminar boundary layer. (details – Assessment sheet B1)



Assumptions

1. steady flow – $\frac{\partial}{\partial t} = 0$
2. flow is largely parallel to the plate –

$$\text{neglect } u_y, \qquad \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}, \qquad \frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$$

3. pressure terms $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$

Using this, it is possible to show that the Navier-Stokes equations become

$$\begin{aligned} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= \nu \frac{\partial^2 u_x}{\partial y^2} \\ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} &= 0 \end{aligned}$$

[NB. The assumption that $\frac{\partial p}{\partial x} = 0$ *is* an assumption. Curved surfaces – later in the course – do not have this restriction.]

To solve these, we combine the two equations. It turns out best to write u_x and u_y in terms of a *stream function*

$$u_x = \frac{\partial \Psi}{\partial y} \qquad u_y = -\frac{\partial \Psi}{\partial x}$$

In turn, the stream function

$$\Psi = \sqrt{\nu x U_\infty} f(\zeta)$$

where

$$\zeta = \frac{y}{\sqrt{\nu x / U_\infty}}$$

In terms of f the governing equations can be written

$$f \frac{d^2 f}{d\zeta^2} + 2 \frac{d^3 f}{d\zeta^3} = 0$$

– a 3rd order ODE. Solve using Runge-Kutta function `rkfixed` in MathCad.

We can use Blasius solution to determine flow conditions in boundary layer :

Determine $\zeta \rightarrow$ find $f, f' \rightarrow$ work back to u.

Note

$$\frac{u_x}{U_\infty} = f'(\zeta), \qquad \frac{u_y}{U_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{U_\infty x}}\,(\zeta f' - f)$$

Remember

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}$$

where τ_0 is the surface stress

$$\tau_0 = \mu\left(\frac{\partial u_x}{\partial y}\right)_{y=0}$$

We can evaluate $\left(\frac{\partial u_x}{\partial y}\right)_{y=0}$ from Blasius,

$$C_f = \frac{2}{\sqrt{\mathcal{R}e_x}}\left(\frac{d^2 f}{d\zeta^2}\right)_{\zeta=0}$$

Since $f'' = 0.332$, this gives

$$C_f = \frac{0.664}{\sqrt{\mathcal{R}e_x}}$$

Summary (A)

- Fluid flow governed by cons. of mass, momentum.
- Draw *control volume* around problem + balance inputs, outputs
  - *Integral* formulation
  - May need to consider small elemental areas $dA = ydx$, $dA = 2\pi rdr$ (polar coordinates)
  - Leads to
  - von Karman method
- Differential formulation – *Navier-Stokes Equations*
  - Solve via computer
  - or simplify to give ODE + boundary conditions

- Required to have $u_{||} = 0$ next to wall – implies *boundary layer* where flow adjusts
- Simplest form : *laminar boundary layer*
- Given flow profile in some form, can work out drag on wall
- Usually express drag as *drag coefficient*

$$C_f = \frac{F/A}{\frac{1}{2}\rho U_\infty^2}$$

  - $C_f = C_f(\mathcal{R}e_x)$
  - Also define $\overline{C_f} = \overline{C_f}(\mathcal{R}e_L)$
  - Various empirical and mathematical (Blasius soln) relations available for C_f, $\overline{C_f}$.