Navier-Stokes Equations – 2d case

SOE3211/2 Fluid Mechanics lecture 3
• conservation of mass, momentum.
• often written as set of pde’s
• differential form – fluid flow at a point
• 2d case, incompressible flow:

Continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

• conservation of mass
• seen before – potential flow
Momentum equations:

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
&\quad + \nu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right] + f_x \\
\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \\
&\quad + \nu \left[ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right] + f_y
\end{align*}
\]

- \(x\) and \(y\) cmpts
- 3 variables, \(u_x, u_y, p\)
- linked equations
- need to simplify by considering details of problem
The NSE are

- Non-linear – terms involving $u_x \frac{\partial u_x}{\partial x}$
- Partial differential equations – $u_x, p$
  functions of $x, y, t$
- 2nd order – highest order derivatives $\frac{\partial^2 u_x}{\partial x^2}$
- Coupled – momentum equation involves $p, u_x, u_y$
The NSE are

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Two ways to solve these equations

1. **Apply to simple cases** – simple geometry, simple conditions
   – and reduce equations until we can solve them
2. **Use computational methods** – CFD (SOE3212/3)
Equation analysis (A)

Consider the various terms:

\[
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right] + f_x
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- change of \( u_x \) at a point
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\]

\[
u \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}
\]

- transport/advection term
- how does flow \((u_x, u_y)\) move \(u_x\)?
- non-linear
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- pressure gradient – usually drives flow
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- viscous term – effect of viscosity \( \nu \) on flow
- has a diffusive effect
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- external body forces – e.g. gravity

\[ f_x \]
Laminar flow between plates (A)

Fully developed laminar flow between infinite plates at $y = \pm a$

What do we expect from the flow?

- $u = 0$ at walls
- Flow symmetric around $y = 0$
- Flow parallel to walls
Navier-Stokes Equations – 2d case

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Flow parallel to walls – we expect

\[ u_y = 0, \quad \frac{dp}{dy} = 0 \quad \text{and} \quad u_x = u_x(y) \]
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Flow fully developed – no change in profile in streamwise direction

i.e. \( \frac{\partial}{\partial t} = 0 \), \( \frac{\partial}{\partial x} = 0 \)
Navier-Stokes Equations – 2d case

\[
\begin{align*}
\frac{\partial}{\partial t} + \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
&+ \nu \left( \frac{\partial^2 u_x}{\partial y^2} \right)
\end{align*}
\]

Flow fully developed – no change in profile in streamwise direction

i.e. \( \frac{\partial}{\partial t} = 0, \quad \frac{\partial}{\partial x} = 0 \)
So momentum equation becomes

\[ 0 = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{d^2 u_x}{dy^2} \]
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Integrate once:

\[ y \frac{dp}{dx} = \rho \nu \frac{du_x}{dy} + C_1 \]
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Integrate again

\[ \frac{1}{2} y^2 \frac{dp}{dx} = \rho \nu u_x + C_2 \]
So momentum equation becomes

\[ 0 = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{d^2 u_x}{dy^2} \]

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Integrate again

\[ \frac{1}{2} y^2 \frac{dp}{dx} = \rho \nu u_x + C_2 \]

But at \( y = \pm a, u_x = 0 \), so

\[ C_2 = \frac{1}{2} a^2 \frac{dp}{dx} \]
Final solution

\[ u_x(y) = \frac{1}{2\rho \nu} \left( y^2 - a^2 \right) \frac{dp}{dx} \]

– equation of a parabola

Also, remember that

\[ \tau = \mu \frac{\partial u_x}{\partial y} \]

So from this we see that in this case

\[ \tau = y \frac{dp}{dx} \]
Flow down inclined plane (A)

– Flow of liquid down inclined plane

Take $x$-component momentum equation

\[
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \nu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right] + f_x
\]
Note:

1. Steady flow
2. $u_x(y)$ only
3. No pressure gradient
4. $f_x = g \sin \alpha$
Note:

1. Steady flow
2. \( u_x(y) \) only
3. \textit{No pressure gradient}
4. \( f_x = g \sin \alpha \)

Equation becomes

\[
\frac{d^2 u_x}{dy^2} = -\frac{g}{\nu} \sin \alpha
\]

which we can integrate easily.
Boundary conditions:

- lower surface \( u_x(0) = 0 \)
- upper surface \( \frac{du_x}{dy} = 0 \)
Boundary conditions:

- lower surface – $u_x(0) = 0$
- upper surface – $\frac{du_x}{dy} = 0$

Solution

$$u_x = \frac{g}{\nu} \sin \alpha \left( hy - \frac{y^2}{2} \right)$$
Tips (A)

Most NSE problems will be time-independent. They will probably only involve one direction of flow, and one coordinate direction. They will probably be either pressure driven (so no viscous term) or shear driven (ie. viscous related, so no pressure term).

Thus, most NSE problems will lead to a 2nd order ODE for a velocity component \((u_x \text{ or } u_y)\) as a function of one coordinate \((x \text{ or } y)\).

Thus we would expect to integrate twice, and to impose two boundary conditions.

- A \textit{wall} boundary condition produces a fixed value: eg. \(u_x = 0\).
- A \textit{free surface} produces a \textit{zero gradient} condition, eg. \(\frac{du_x}{dy} = 0\).