NSE – Integral form

SOE3211/2 Fluid Mechanics lecture 2

Recap

Fluid flows governed by conservation of mass, momentum. We can use this to solve flow problems.

Draw box (*control volume*) around region of interest, then equate mass flux into, out of region.

– Integral formulation of NSE

Momentum equation (A)

We can also write the conservation of momentum in a similar form.

The momentum of a small piece of fluid will be $\rho \underline{u} dV$. So the rate of change is

$$\frac{d}{dt} \iiint_V \rho \underline{u} dV$$

What is the flux of momentum?

Momentum equation (A)

We can also write the conservation of momentum in a similar form.

The momentum of a small piece of fluid will be $\rho \underline{u} dV$. So the rate of change is

$$\frac{d}{dt} \iiint_V \rho \underline{u} dV$$

What is the flux of momentum? In fact it is $(\rho \underline{u}) \underline{u} . d\underline{A}$ through a bit of area $d\underline{A}$.

Momentum equation (A)

We can also write the conservation of momentum in a similar form.

The momentum of a small piece of fluid will be $\rho \underline{u} dV$. So the rate of change is

$$\frac{d}{dt} \iiint_V \rho \underline{u} dV$$

What is the flux of momentum? In fact it is $(\rho \underline{u}) \underline{u}.d\underline{A}$ through a bit of area $d\underline{A}$.

Thus we can write

$$\frac{d}{dt} \iiint_V \rho \underline{u} dV + \iint_S (\rho \underline{u}) \underline{u}.d\underline{A} = \sum \text{Forces}$$

The forces are

1. body forces, eg. gravity, and
2. surface forces – pressure, viscous stress, etc.

2. can be written as a stress $\underline{\underline{\tau}}$, and so

$$\frac{d}{dt}\iiint_V \rho \underline{u} dV + \iint_S (\rho \underline{u})\underline{u}.d\underline{A} = \iint_S \underline{\underline{\tau}}.d\underline{A}$$

Can we make use of this?

The forces are

1. body forces, eg. gravity, and
2. surface forces – pressure, viscous stress, etc.

2. can be written as a stress $\underline{\underline{\tau}}$, and so

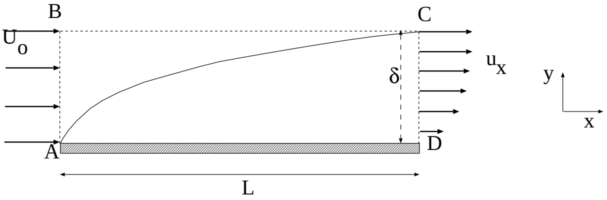
$$\iint_S (\rho \underline{u})\underline{u}.d\underline{A} = \iint_S \underline{\underline{\tau}}.d\underline{A}$$

Can we make use of this? If we assume that the flow is steady, i.e.

$$\frac{d}{dt}\iiint_V \rho \underline{u} dV = 0$$

and choose our *control volume* V intelligently, then we can use this to calculate the forces on a body.

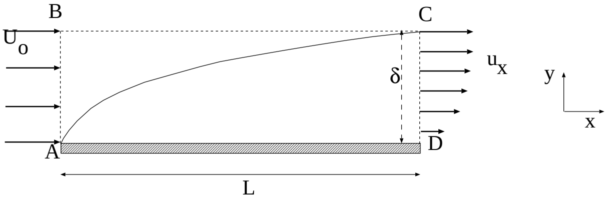
B.L. flow (again) (B)



Assume $\quad u_x = U_0 \sin \dfrac{\pi y}{2\delta} \quad$ across CD

What happens if we consider momentum fluxes?

B.L. flow (again) (B)



Assume $u_x = U_0 \sin \frac{\pi y}{2\delta}$ across CD

What happens if we consider momentum fluxes?

In the x direction : Momentum flux AB

$$\mathcal{F}_{AB} = (\rho U_0) \times U_0 \times (\delta \times W) = \rho U_0^2 \delta W$$

Momentum flux CD : For a small element dy the momentum flux is

$$(\rho u_x)u_x(W \times dy)$$

Momentum flux CD : For a small element dy the momentum flux is

$$(\rho u_x)u_x(W \times dy)$$

so integrating this

$$\begin{aligned}
\mathcal{F}_{CD} &= W \times \int_0^{\delta} \rho u_x^2 dy \\
&= \rho W \int_0^{\delta} U_0^2 \sin^2 \frac{\pi y}{2\delta} dy \\
&= \rho U_0^2 W \int_0^{\delta} \frac{1}{2} \left(1 - \cos \frac{\pi y}{\delta}\right) dy \\
&= \frac{\rho U_0^2 W}{2} \left[y - \frac{\delta}{\pi} \sin \frac{\pi y}{\delta} \right]_0^{\delta} \\
&= \frac{\rho U_0^2 \delta W}{2}
\end{aligned}$$

Momentum flux BC : We can guess that the fluid flowing out of BC shares the undisturbed flow velocity in the x direction. Hence

$$\mathcal{F}_{BC} = (\text{mass flux})_{BC} \times U_0 = \rho U_0^2 \delta W \left[1 - \frac{2}{\pi}\right]$$

Momentum flux BC : We can guess that the fluid flowing out of BC shares the undisturbed flow velocity in the x direction. Hence

$$\mathcal{F}_{BC} = (\text{mass flux})_{BC} \times U_0 = \rho U_0^2 \delta W \left[1 - \frac{2}{\pi}\right]$$

Thus

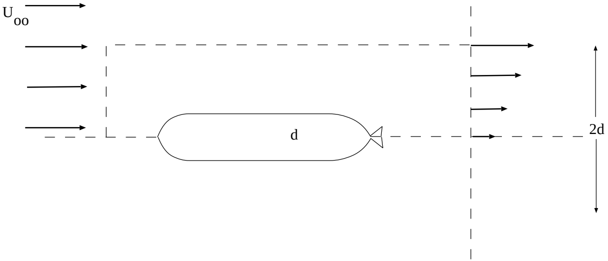
$$
\begin{aligned}
(\text{Net momentum flux})_x &= \mathcal{F}_{in} - \mathcal{F}_{out} \\
&= \mathcal{F}_{AB} - \mathcal{F}_{CD} - \mathcal{F}_{BC} \\
&= \rho U_0^2 \delta W \left[1 - \frac{1}{2} - \left(1 - \frac{2}{\pi}\right)\right] \\
&= \rho U_0^2 \delta W \left(\frac{2}{\pi} - \frac{1}{2}\right)
\end{aligned}
$$

The only surface left is AD. There is no fluid flowing across this surface, so this must represent the force exerted on the plate AD by the fluid flow.

NB. We have implicitly assumed there are no viscous stresses of importance on AB, BC, CD.

Worked example (B)

Tests of a model underwater projectile in a water tunnel show that the velocity profile in a certain cross-section of the wake may be approximated to the shape of a cone. At this section the centreline velocity is equal to half the free stream velocity and the width of the wake is equal to twice the missile diameter. Use the von Karman integral analysis to estimate the drag coefficient of the torpedo.



The wake velocity is

$$\frac{u}{u_\infty} = \frac{1}{2}\left(1 + \frac{r}{d}\right)$$

We work in cylindrical polar coordinates, so over the ends $dA = 2\pi r dr$.

Consider mass fluxes first :

$$\phi^{(\rho)}_{front} \quad = \quad \phi^{(\rho)}_{sides} + \phi^{(\rho)}_{back}$$

The wake velocity is

$$\frac{u}{u_\infty} = \frac{1}{2}\left(1 + \frac{r}{d}\right)$$

We work in cylindrical polar coordinates, so over the ends $dA = 2\pi r dr$.

Consider mass fluxes first :

$$\begin{aligned} \phi^{(\rho)}_{front} &= \phi^{(\rho)}_{sides} + \phi^{(\rho)}_{back} \\ \text{i.e.} \qquad \int_0^d \rho U_\infty 2\pi r dr &= \phi^{(\rho)}_{sides} + \int_0^d \rho u 2\pi r dr \end{aligned}$$

The wake velocity is

$$\frac{u}{u_\infty} = \frac{1}{2}\left(1 + \frac{r}{d}\right)$$

We work in cylindrical polar coordinates, so over the ends $dA = 2\pi r dr$.

Consider mass fluxes first :

$$\begin{aligned} \phi^{(\rho)}_{front} &= \phi^{(\rho)}_{sides} + \phi^{(\rho)}_{back} \\ \text{i.e.} \quad \int_0^d \rho U_\infty 2\pi r dr &= \phi^{(\rho)}_{sides} + \int_0^d \rho u 2\pi r dr \end{aligned}$$

Rearanging,

$$\phi^{(\rho)}_{sides} = \int_0^d \rho(U_\infty - u) 2\pi r dr$$

Momentum balance over control volume :

$$\begin{aligned} \text{Force} &= \mathcal{F}_{front} - \mathcal{F}_{sides} - \mathcal{F}_{back} \\ &= \int_0^d \rho U_\infty^2 2\pi r dr - U_\infty \phi_{sides}^{(\rho)} - \int_0^d \rho u^2 2\pi r dr \end{aligned}$$

Momentum balance over control volume :

$$\begin{aligned} \text{Force} &= \mathcal{F}_{front} - \mathcal{F}_{sides} - \mathcal{F}_{back} \\ &= \int_0^d \rho U_\infty^2 2\pi r dr - U_\infty \phi_{sides}^{(\rho)} - \int_0^d \rho u^2 2\pi r dr \end{aligned}$$

Substituting for $\phi_{sides}^{(\rho)}$ and rearanging, we get

$$F_D = \int_0^d \rho u \left(U_\infty - u \right) 2\pi r dr$$

Momentum balance over control volume :

$$
\begin{aligned}
\text{Force} &= \mathcal{F}_{front} - \mathcal{F}_{sides} - \mathcal{F}_{back} \\
&= \int_0^d \rho U_\infty^2 2\pi r dr - U_\infty \phi_{sides}^{(\rho)} - \int_0^d \rho u^2 2\pi r dr
\end{aligned}
$$

Substituting for $\phi_{sides}^{(\rho)}$ and rearanging, we get

$$
F_D = \int_0^d \rho u \left(U_\infty - u \right) 2\pi r dr
$$

Now substituting for u we have

$$
\begin{aligned}
F_D &= \frac{\pi \rho U_\infty^2}{2} \int_0^d \left[1 - \left(\frac{r}{d} \right)^2 \right] r dr \\
&= \frac{\pi \rho U_\infty^2}{2} \left[\frac{r^2}{2} - \frac{r^4}{4d^2} \right]_0^d = \frac{\pi \rho U_\infty^2}{2} \left(\frac{d^2}{4} \right)
\end{aligned}
$$

Finally, the drag coefficient is

$$C_d = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 A_M}$$

where A_M is the frontal area of the projectile $= \pi d^2/4$.

So in this case $C_d = 1$

This is often known as the *von Karman integral formulation*

von Karman integral formulation (A)

- Measure wake velocities
- Draw apprpriate control volume
- Apply integral forms of mass, momentum equations
- Ignore viscous stresses

von Karman integral formulation (A)

- Measure wake velocities
- Draw apprpriate control volume
- Apply integral forms of mass, momentum equations
- Ignore viscous stresses

Mathematics equivalent to integrating *momentum deficit* over area of wake.

In cylindrical coordinates

$$F_z = \int \rho u_z \left(U_\infty - u_z\right) 2\pi r dr$$

In cartesian coordinates

$$\frac{F_x}{W} = \int \rho u_x \left(U_\infty - u_x\right) dy$$

Other fluxes (B)

So far we have defined a volume flux

$$\phi_V = \underline{u}.d\underline{A}$$

and a mass flux

$$\phi_\rho = \rho\underline{u}.d\underline{A}$$

and a momentum flux

$$\phi_m = (\rho\underline{u})\ \underline{u}.d\underline{A}$$

Other fluxes (B)

So far we have defined a volume flux

$$\phi_V = \underline{u}.d\underline{A}$$

and a mass flux

$$\phi_\rho = \rho\underline{u}.d\underline{A}$$

and a momentum flux

$$\phi_m = (\rho\underline{u})\ \underline{u}.d\underline{A}$$

Define fluxes for any quantity of interest :

Kinetic energy $\quad \phi_{ke} = \left(\frac{1}{2}\rho u^2\right)\ \underline{u}.d\underline{A}$

Angular mom. $\quad \phi_{am} = (\rho\underline{r} \times \underline{u})\,\underline{u}.d\underline{A}$