Laminar Boundary Layers

SOE3211/2 Fluid Mechanics lecture 4

4.1 Boundary Layer (A)

Boundary conditions for flow at a wall

$$\underline{u}_{rel} = \underline{0}$$

– the flow shares the velocity of the wall.

For a stationary wall, $\underline{u} = 0$

Thus, no matter what the flow is doing anywhere else, there must be a laminar region somewhere close to the wall.

The near wall region where the flow adapts to $\underline{u}_{rel} = \underline{0}$ is called the $\boxed{\text{Boundary Layer}}$

(Almost) everything of importance happens in the boundary layer – important effects on

- lift
- drag
- heat transfer

$\Rightarrow$ going to spend quite some time on it.

In particular, want to determine *wall shear stress.*

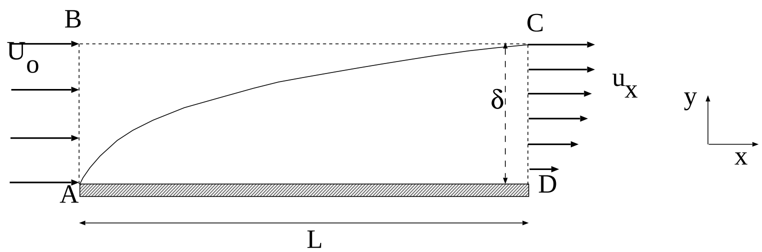
$$\tau_0 = \frac{F_0}{A}$$

Relate to friction coefficient

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U^2}$$

4.2 Laminar Boundary Layer (A)

Laminar flow around a flat plate. Fluid flows past the plate with velocity U_∞



Effect of the plate propagates outwards

$\Rightarrow$ broadening 'region of influence' around the plate.

We can define a distance δ – the *boundary layer thickness* – which is the distance at which the flow velocity begins to drop

Define δ as the distance that

$$u_x(\delta) = 99\% U_\infty$$

(the factor 99% is somewhat arbitary).

Flow is largely parallel to the plate except in 'adjustment region' around δ

Other definitions of δ possible

4.3 Skin Friction Coefficient C_f (A)

Define coefficient

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}$$

where τ_0 is the surface stress (force per unit area of the surface).

Useful : if we have C_f, can find force on the plate.

Blasius solution to NSE for laminar b.l. flow across plate gives τ_0, so

$$C_f = \frac{0.664}{\sqrt{\mathcal{R}e_x}}$$

with

$$\mathcal{R}e_x = \frac{U_0 x}{\nu}$$

and x is distance along plate. This is a *local* coefficient.

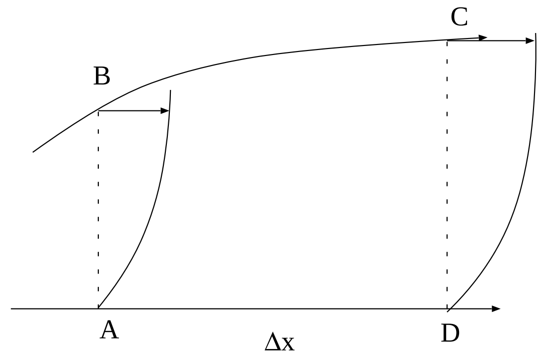
We want to be able to find the drag on a complete plate. Integrate along x, plate length L we get

$$\overline{C_f} = \frac{1.33}{\sqrt{\mathcal{R}e_L}} \quad \text{with} \quad \mathcal{R}e_L = \frac{U_0 L}{\nu}$$

Also : thickness of b.l.

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathcal{R}e_x}}$$

4.4 von Karman analysis (B)



Apply momentum equation →

$$-\tau_0 \Delta x = \int_0^{\delta_2} \rho u_2^2 dy - \int_0^{\delta_1} \rho u_1^2 dy - \rho U_0^2 (\delta_2 - \delta_1)$$

In order :

$-\tau_0 \Delta x$ shear force opposing motion

$\int_0^{\delta_2} \rho u_2^2 dy$ rate of momentum transfer through CD

$-\int_0^{\delta_1} \rho u_1^2 dy$ same through AB

$-\rho U_0^2 (\delta_2 - \delta_1)$ momentum through BC

But

$$U_0(\delta_2 - \delta_1) = \int_0^{\delta_2} u_2 dy - \int_0^{\delta_1} u_1 dy$$

Rearanging we find

$$-\tau_0 \Delta x = \rho U_0^2 \left[\int_0^{\delta_2} \left(\left(\frac{u_2}{U_0} \right)^2 - \frac{u_2}{U_0} \right) dy - \int_0^{\delta_1} \left(\left(\frac{u_1}{U_0} \right)^2 - \frac{u_1}{U_0} \right) dy \right]$$

Limit as $\Delta x \to 0$:

$$\tau_0 = \rho U_0^2 \frac{d}{dx} \int_0^{\delta} \left(\left(\frac{u}{U_0} \right)^2 - \frac{u}{U_0} \right) dy$$

or

$$\tau_0 = \rho U_0^2 \frac{d\theta}{dx}$$

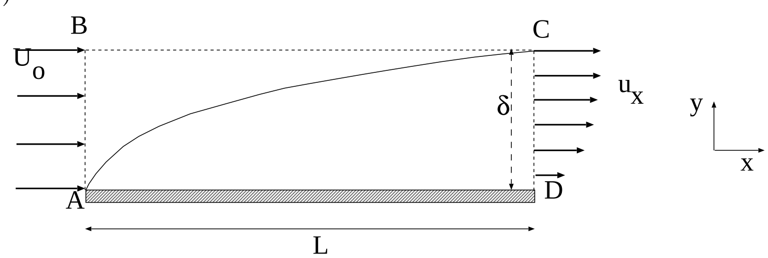
where

$$\theta = \int_0^\infty \left(1 - \frac{u_x}{U_0}\right) \frac{u_x}{U_0} dy$$

Importance of this? If we know θ or u (measurements, theory), can calculate forces.

4.5 Blasius Solution (B)

Mathematical solution for laminar boundary layer. (details – Assessment sheet B1)



Assumptions

1. steady flow – $\frac{\partial}{\partial t} = 0$
2. flow is largely parallel to the plate –

$$\text{neglect } u_y, \qquad \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}, \qquad \frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$$

3. pressure terms $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$

Using this, it is possible to show that the Navier-Stokes equations become

$$\begin{aligned} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= \nu \frac{\partial^2 u_x}{\partial y^2} \\ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} &= 0 \end{aligned}$$

[NB. The assumption that $\frac{\partial p}{\partial x} = 0$ *is* an assumption. Curved surfaces – later in the course – do not have this restriction.]

To solve these, we combine the two equations. It turns out best to write u_x and u_y in terms of a *stream function*

$$u_x = \frac{\partial \Psi}{\partial y} \qquad u_y = -\frac{\partial \Psi}{\partial x}$$

In turn, the stream function

$$\Psi = \sqrt{\nu x U_\infty} f(\zeta)$$

where

$$\zeta = \frac{y}{\sqrt{\nu x / U_\infty}}$$

In terms of f the governing equations can be written

$$f \frac{d^2 f}{d\zeta^2} + 2 \frac{d^3 f}{d\zeta^3} = 0$$

– a 3rd order ODE. Solve using Runge-Kutta function `rkfixed` in MathCad.

We can use Blasius solution to determine flow conditions in boundary layer :

Determine $\zeta \to$ find $f, f' \to$ work back to u.
Note

$$\frac{u_x}{U_\infty} = f'(\zeta), \qquad \frac{u_y}{U_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{U_\infty x}}\,(\zeta f' - f)$$

Remember

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}$$

where τ_0 is the surface stress

$$\tau_0 = \mu \left(\frac{\partial u_x}{\partial y} \right)_{y=0}$$

We can evaluate $\left(\frac{\partial u_x}{\partial y}\right)_{y=0}$ from Blasius,

$$C_f = \frac{2}{\sqrt{\mathcal{R}e_x}} \left(\frac{d^2 f}{d\zeta^2} \right)_{\zeta=0}$$

Since $f'' = 0.332$, this gives

$$C_f = \frac{0.664}{\sqrt{\mathcal{R}e_x}}$$

4.6 Summary (A)

- Fluid flow governed by cons. of mass, momentum.
- Draw *control volume* around problem + balance inputs, outputs
  - *Integral* formulation
  - May need to consider small elemental areas $dA = ydx$, $dA = 2\pi rdr$ (polar coordinates)
  - Leads to
  - von Karman method
- Differential formulation – *Navier-Stokes Equations*
  - Solve via computer
  - or simplify to give ODE + boundary conditions
- Required to have $u_{\|} = 0$ next to wall – implies *boundary layer* where flow adjusts
- Simplest form : *laminar boundary layer*
- Given flow profile in some form, can work out drag on wall
- Usually express drag as *drag coefficient*

$$C_f = \frac{F/A}{\frac{1}{2}\rho U_\infty^2}$$

  - $C_f = C_f(\mathcal{R}e_x)$
  - Also define $\overline{C_f} = \overline{C_f}(\mathcal{R}e_L)$
  - Various empirical and mathematical (Blasius soln) relations available for C_f, $\overline{C_f}$.