Solutions (2002)

Question 1. a. Equating the two expressions for τ gives a first order ode :

$$\frac{du(r)}{dr} = \frac{r}{2\mu}\frac{dp}{dz}$$

Integrating this, and imposing the boundary condition that the velocity $u(r)$ must be zero at the wall ($r = R$) gives the required profile

$$u(r) = -\frac{dp}{dz}\frac{1}{4\mu}\left(R^2 - r^2\right) \tag{1}$$

b. To get Q we need to integrate $u(r)$ over the end of the pipe. Thus :

$$Q = \int_0^R u(r)2\pi r dr = -\frac{dp}{dz}\frac{2\pi}{4\mu}\int_0^R \left(R^2 - r^2\right) r dr = -\frac{dp}{dz}\frac{\pi}{8\mu}R^4$$

c. Dividing this last expression by πR^2 gives

$$\frac{Q}{\pi R^2} = \overline{u} = -\frac{dp}{dz}\frac{R^2}{8\mu}$$

Dividing (1) by this

$$\begin{aligned}\frac{u(r)}{\overline{u}} &= \frac{-\frac{dp}{dz}\frac{1}{4\mu}\left(R^2 - r^2\right)}{-\frac{dp}{dz}\frac{R^2}{8\mu}} \\ &= 2\frac{\left(R^2 - r^2\right)}{R^2}\end{aligned}$$

Rearanging this gives the expression in the question.

$$u(r) = 2\overline{u}\frac{R^2 - r^2}{R^2}$$

d. The laminar flow is smooth : neighbouring streamlines remain close, and the profile is parabolic. At $\mathcal{R}e = 8000$ the flow will have become turbulent. Turbulent flows are characterised by a chaotic component of the flow superimposed on whatever mean flow is present. Neighbouring streamlines diverge rapidly. The mean flow profile will become flatter at the centre of the pipe and steeper at the edges, characterised by a 1/7 power law.

Question 2. Solution.

a. Standard energy spectrum diagram

b. Flow behind a cylinder demonstrates different characteristics at different Reynolds numbers as it transitions from smooth, laminar flow at $\mathcal{R}e < 1$ to a turbulent wake for high Reynolds numbers. At $\mathcal{R}e \sim 100$ the flow pattern takes the form of two stable recirculation zones behind the cylinder. As the Reynolds number is increased from this these zones become unstable, and eventually are shed alternating from either side, to be advected downstream by the mean flow. This row of alternating vortices is called the von Karman vortex street. A sketch may be included in this explanation.

The frequency of shedding is characterised by a dimensionless group called the Strouhal number.

c. The sphere will reach terminal velocity where the forces, including hydrodynamic forces, are in equilibrium. The three forces involved are i. the weight of the sphere, ii. the fluid upthrust and iii. the hydrodynamic drag. Balancing these gives

$$W - U = D$$

where

$$\begin{aligned} W &= \frac{4}{3}\pi r^3 \rho_{steel} \quad \text{is the weight of the ball} \\ U &= \frac{4}{3}\pi r^3 \rho_{water} \quad \text{is the upthrust} \\ D &= \frac{1}{2}\rho_{water} V^2 C_D(\mathcal{R}e) A \quad \text{is the drag force } (A = \text{area of sphere}) \end{aligned}$$

Since $D = D(V)$, the velocity, (non-linearly through the drag coefficient), we will need to guess the velocity and iterate to find a solution.

$$W - U = \frac{4}{3}\pi r^3 \left(\rho_{steel} - \rho_{water}\right) = \frac{1}{2}\rho_{water} V^2 C_D \pi r^2$$

Rearanging this and substituting values $18.18r = V^2 C_D$ Also $\mathcal{R}e = \frac{Vd}{\nu} = 3922V$ For each case, we substitute the value of r, then guess a starting velocity, find $\mathcal{R}e$, C_D (read from the graph) and V^2, then iterate this until V no longer changes.

d. Golf balls travel in a regime close to the 'drag crisis', where the boundary layer can become turbulent, and the drag coefficient and hence drag force can drop dramatically. The dimples help to force the boundary layer into turbulence.

Question 3. Solution

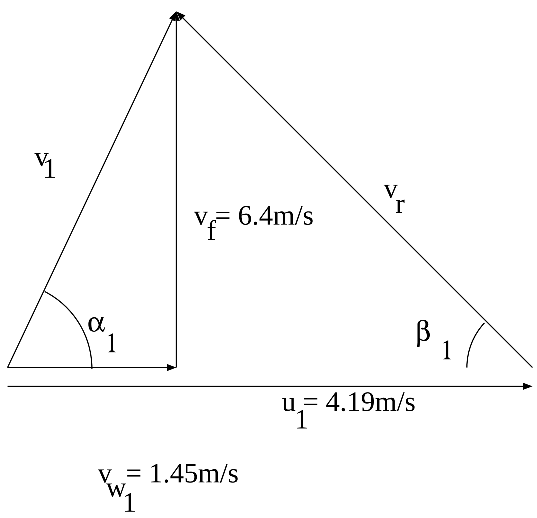
a.i. No shock condition implies the fluid meets or leaves the impeller blades tangentially, ie $\beta' = \beta$.

ii. Hydraulic losses are the fluid flow losses (as against frictional losses in bearings etc) – basically the sum of impeller loss (shocks) leakage loss (fluid missing the impeller) and casing loss (fluid drag against the casing).

b. At the inlet :

$$\begin{aligned}
\text{Inlet area } A_1 &= \pi d_1 h_1 = \pi \times 0.4 \times 0.1 = 0.126 \text{ m}^2 \\
\text{Inlet flow through impeller } v_{f1} &= \frac{Q}{A_1} = 6.366 \text{ m/s} \\
\text{Whirl velocity } v_{w1} &= 1.45 \text{ m/s} \\
\text{Rotational speed } N &= 200\text{RPM} = 3.33 \text{ rps} \quad \Rightarrow \quad \omega = 20.94\ rad/\text{s} \\
\text{Rotational velocity } u_1 &= r_1\omega = 4.189 \text{ m/s}
\end{aligned}$$

From this information we can sketch the inlet triangle :



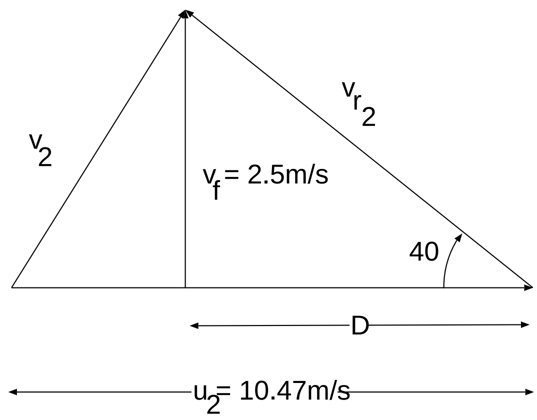
From this the absolute and relative velocities can be evaulated straightforwardly, as can α.

c. The angle β on the figure above can be evaluated thus :

$$\beta = \arctan\frac{v_{f1}}{u_1 - v_{w1}} = \arctan\frac{6.37}{4.19 - 1.45} = 66.7°$$

For the no-shock condition the water should meet the blade tangentially. To achieve this the blade should be set at this angle of 66.7°.

d. At the outlet, the area is now 0.314 m^2, so $v_{f_2} = 2.546$ m/s. The rotational velocity $u_2 = 10.47$ m/s. We can draw the outlet triangle thus :



and so the whirl velocity at the outlet is $v_w = u_2 - \Delta = 7.49$ m/s Thus

$$v_2 = \sqrt{v_{f2}^2 + v_{w2}^2} = \sqrt{2.5^2 + 7.49^2} = 7.9 \text{ m/s}$$

e. The Euler equation is

$$H_{imp} = \frac{P}{\dot{m}g} = \frac{1}{g}\left(u_2 v_{w2} - u_1 v_{w1}\right)$$

Evaluating this

$$H_{imp} = \frac{1}{9.81}\left(10.47 \times 7.49 - 4.19 \times 1.45\right) = 7.37 \text{ metres of water}$$

Question 4. Solution.

a. The curve is actually a straight line giving a lift coefficient of 0.6 for $\alpha = 0$ and 1.3 for $\alpha = 5°$. This is not uncommon (small angle approximation). Beyond $\alpha = 5°$ the curve may stop being linear, and will definitely reach a maximum and drop off. The maximum is the stall angle, and is caused by vortex shedding from the leading edge of the airfoil.

b. The force exerted on an airfoil is

$$F_L = \frac{1}{2}\rho U^2 A C_L$$

with A the plan area of the airfoil. However the different pieces of the rotor will be moving at differing speed. So we will consider a section of the blade $dA = cdr$:

$$dF = \frac{1}{2}\rho\Omega^2 c C_L r^2 dr$$

For 4 blades the total force is therefore

$$F_T = \int_0^R 4dF = \frac{1}{2}\left(4\rho\Omega^2 c\right)\int_0^R C_L r^2 dr$$

Substituting $y = r/R$, $dy = dr/R$,

$$F_T = \frac{1}{2}\left(4\rho\Omega^2 cR^3\right)\int_0^1 C_L y^2 dy$$

This is in the desired form, with

$$q = \left(4\rho\Omega^2 cR^3\right)$$

c. The angle of attack is

$$\alpha(r) = \alpha_0\left(1 - \frac{r}{2R}\right), \qquad \alpha_0 = 5^\circ$$

The lift varies as

$$C_L(\alpha) = 0.6\,(1 + 0.233\alpha) \qquad (\alpha \text{ in degrees})$$

Doing the integral :

$$\int_0^1 C_L y^2 dy = \int_0^1 (1.299 - 0.3495y)y^2 dy = 0.3456$$

The force on the helicopter is 7651.8N. Rearanging, this gives

$$q = 2\frac{F_T}{0.3456} = 44281.25$$

Thus

$$\Omega^2 = \frac{q}{4\rho cR^3} = \frac{44281.25}{4 \times 1.2 \times 0.12 \times 6^3} = 355.9$$

This gives $\Omega = 18.87$/s, or 180RPM.

Question 5. Solution

a.i.

$$\begin{aligned}\nabla.\underline{u} &= \frac{\partial}{\partial x}(Ax\sin\omega t) + \frac{\partial}{\partial y}(-Ay\sin\omega t)\\ &= (A\sin\omega t) - (A\sin\omega t)\\ &= 0\end{aligned}$$

a.ii. For an incompressible flow, $\nabla.\underline{u} = 0$. This is satisfied, so this velocity could be that for incompressible flow.

b.i. Continuity equation :

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Since the flow is fully developed, $\frac{\partial u_x}{\partial x} = 0$, so $\frac{\partial u_y}{\partial y} = 0$. Integrating this, u_y =const. At the boundaries we have the result $u_y = V_W$, so the velocity u_y must take this value throughout.

ii. The x-component of the NSE is

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right] + f_x$$

There are no body forces, so $f_x = 0$. The flow is not time-dependent, so $\frac{\partial}{\partial t}$ terms are zero. The flow is fully developed, so the flow profile does not change with x, so derivatives w.r.t. this variable are also zero. This leaves

$$u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}$$

u_x is a function of y alone, so the derivatives become ordinary. We now have expressions for u_y and $\frac{\partial p}{\partial x}$: substituting these in gives

$$V_W \frac{du_x}{dy} = \frac{K}{\rho} + \nu \frac{d^2 u_x}{dy^2}$$

Rearanging this gives the result in the question.

iii. Solving this equation is quite challenging : there are two possible approaches. One is to regard this as a linear inhomogeneous 2nd order equation and look for a particular solution. The method hinted at in the question is to make the substitution $X = \frac{du_x}{dy}$, which results in the 1st order equation

$$\frac{dX}{dy} - \frac{V_W}{\nu} X = -\frac{K}{\nu \rho}$$

Thus

$$\int \frac{dX}{\frac{V_W}{\nu} X - \frac{K}{\nu \rho}} = \int dy = y + C \tag{2}$$

The lhs of this expression is in the form given in the question, with

$$a = \frac{V_W}{\nu} \quad \text{and} \quad b = -\frac{K}{\nu \rho}$$

Thus equation (2) becomes

$$\frac{1}{a}\log(aX + b) = y + C$$
$$\Rightarrow a\frac{du_x}{dy} = A\mathrm{e}^{ay} - b$$

(constants A and C are basically the same constant, just rewritten appropriately). We can integrate this again

$$au_x = \frac{A}{a}\mathrm{e}^{ay} - by + B$$

Multiplying through by a may simplify things

$$a^2 u_x = A\mathrm{e}^{ay} - aby + B$$

(redefining B again). Now the constants A and B can be evaluated from the boundary conditions. $u_x = 0$ at $y = 0$ and $y = h$, so

$$A = \frac{abh}{\mathrm{e}^{ah} - 1}, \qquad B = \frac{-abh}{\mathrm{e}^{ah} - 1}$$

and

$$\Rightarrow u_x = \frac{bh}{a}\left(\frac{\mathrm{e}^{ay} - 1}{\mathrm{e}^{ah} - 1}\right) - \frac{by}{a}$$

Writing

$$\frac{b}{a} = -\frac{K}{V_W \rho}$$

gives

$$u_x = -\frac{Kh}{V_W \rho}\left(\frac{\mathrm{e}^{V_W y/\nu} - 1}{\mathrm{e}^{V_W h \nu} - 1}\right) + \frac{Ky}{V_W \rho}$$

which can also be written as

$$u_x = \frac{Kh}{V_W \rho}\left(\frac{y}{h} - \frac{\mathrm{e}^{V_W y/\nu} - 1}{\mathrm{e}^{V_W h \nu} - 1}\right)$$

Question 6. Solution

a.i. Dimensions :

$$\begin{array}{rl} (gH) & [L^2T^{-2}] \\ N & [T^{-1}] \\ D & [L] \\ \rho & [ML^{-3}] \end{array}$$

Writing the expression as

$$(gH)^a N^b D^c \rho^d = [L^2T^{-2}]^a [T^{-1}]^b [L]^c [ML^{-3}]^d$$

we want to find $a - d$ that make this dimensionless. Working through gives

$$\Pi_0 = \frac{gH}{N^2 D^2}$$

as the first dimensionless group.

a.ii. Dimensions of Q are $[L^3 T^{-1}]$, so

$$\Pi_1 = \frac{Q}{ND^3}$$

Dimensions of μ are $[ML^{-1}T^{-1}]$, and $c = -2$

$$\Pi_2 = \frac{\mu}{ND^2\rho}$$

which is a Reynolds number of sorts. Buckingham-Π theorem requires that Π_0 can be written as a function of the others, in other words

$$\frac{gH}{N^2 D^2} = \mathcal{F}\left(\frac{Q}{ND^3}, \frac{\mu}{ND^2\rho}\right)$$

a.iii. We now have 2 extra variables. One, the roughness, might be dimensionless already, but if not we can create a new group

$$\Pi_3 = \frac{\eta}{D}$$

The other will be based on the compressibility K. This has units of Pa, ie. dimensions of pressure, and so $[K] = [ML^{-1}T^{-2}]$. Using the repeating variables from before,

$$\Pi_4 = \frac{K}{N^2 D^2 \rho}$$

These can just be inserted into the previous expression :

$$\frac{gH}{N^2 D^2} = \mathcal{F}\left(\frac{Q}{ND^3}, \frac{\mu}{ND^2\rho}, \frac{K}{N^2 D^2 \rho}, \frac{\eta}{D}\right)$$

b. In vector notation the momentum equation is written

$$\rho\frac{D\underline{u}}{Dt} = -\nabla p + \mu\nabla^2\underline{u} + \underline{F}$$

(from the equation sheet). Dividing through by the density

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u} + \underline{f}$$

Here $\underline{f}$ is the body force per unit volume, which here is the accelleration due to gravity $\underline{g}$.

We need to find dimensionless versions of each of the terms here. The velocity becomes

$$\underline{u}^* = \frac{u}{U_0} \qquad \text{and the time} \quad t^* = \frac{t}{L_0/U_0} = \frac{U_0 t}{L_0}$$

where starred variables represent dimensionless quantities. Using these the first term is

$$\frac{D\underline{u}}{Dt} = \frac{U_0}{L_0}\frac{D^*}{Dt^*}(U_0\underline{u}^*) = \frac{U_0^2}{L_0}\frac{D^*\underline{u}^*}{Dt^*}$$

Working through the other terms in a similar manner we get

$$\frac{U_0^2}{L_0}\frac{D^*\underline{u}^*}{Dt^*} = -\frac{1}{\rho^*}\frac{U_0^2}{L_0}\nabla^* p^* + \nu\frac{U_0}{L_0^2}\nabla^{*2}\underline{u}^* + g$$

Dividing through by $\frac{U_0^2}{L_0}$ gives

$$\frac{D^*\underline{u}^*}{Dt^*} = -\frac{1}{\rho^*}\nabla^* p^* + \frac{\nu}{U_0 L_0}\nabla^{*2}\underline{u}^* + \frac{gL_0}{U_0^2}$$

This is now dimensionless. We can identify

$$\frac{\nu}{U_0 L_0} = \frac{1}{\mathcal{R}e}$$

as the Reynolds number, a measure of the relative importance of the viscous term, and

$$\frac{gL_0}{U_0^2} = \frac{1}{\mathcal{F}r}$$

as the Froude number, a measure of the relative importance of gravitational effects.