Torsion

Reading : Chou & Pagano 6.1,6.2, 6.5

VII.1 Basics

We want to find solutions for cases where a bar of arbitary cross section has been twisted. Assume that the bar is orientated along the z axis, and that the distortion consists of

1. rotation of cross-sections of the bar, with the angle of twist per unit length being constant, plus
2. warping (displacement along the z axis) which is the same for all cross-sections, ie. is independent of z.

Having made these assumptions we will show that the equations of equilibrium and boundary conditions can be satisfied, which will prove that the assumptions are correct.

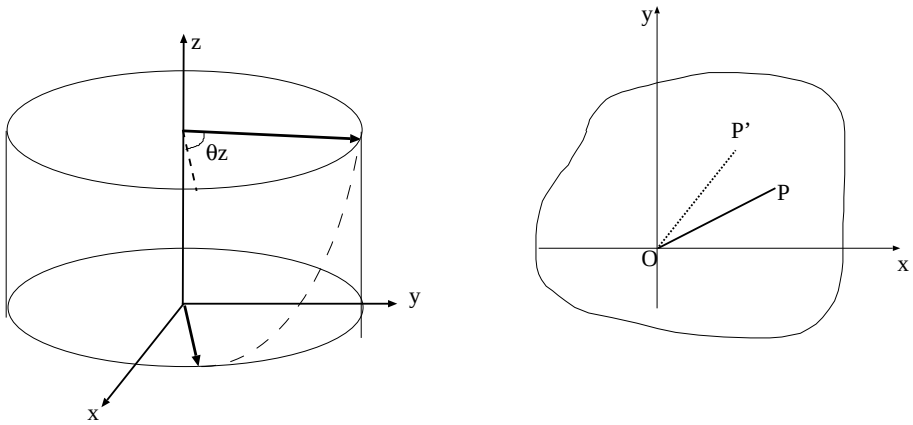

Figure 1: Bar under torsion

The bar being twisted is shown in figure 1. The displacement of any point P can be described as a rotation around a centre of twist O, this being the point in the plane (line in 3d) which is not moved during the deformation. We can simplify the problem by making this the z axis, and taking the plane $z = 0$ to be the plane of zero rotation. If the twist per unit length is a constant, and the angles are small, then the displacements u and v are

$$u = -\alpha z y, \qquad v = \alpha z x \tag{VII.1}$$

The warp is a function of x and y alone, which we write as

$$w = \alpha \psi(x, y) \tag{VII.2}$$

Using these we can evaluate the strain components :

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0$$

$$\gamma_{xz} = \alpha \left(\frac{\partial \psi}{\partial x} - y \right)$$
$$\gamma_{yz} = \alpha \left(\frac{\partial \psi}{\partial y} + x \right)$$

and the corresponding stress components

$$\begin{aligned} \sigma_x = \sigma_y = \sigma_z = \tau_{xy} &= 0 \\ \tau_{xz} &= G\alpha \left(\frac{\partial \psi}{\partial x} - y \right) \\ \tau_{yz} &= G\alpha \left(\frac{\partial \psi}{\partial y} + x \right) \end{aligned} \tag{VII.3}$$

If we substitute these relationships into the equations of equilibrium (V.11), assuming no body forces, we find

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{VII.4}$$

[**Q.VII.1**] By differentiating τ_{xz} with respect to y and τ_{yz} w.r.t x, derive a compatibility equation for these stresses.
[**Q.VII.2**] Show that ψ satisfies Laplace's equation $\nabla^2 \psi = 0$

VII.2 Prandtl Stress Function

Let us look for a stress function $\varphi(x, y)$ for this problem. Equation (VII.4) is satisfied if

$$\tau_{xz} = \frac{\partial \varphi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \varphi}{\partial x}$$

Substituting these in (VII.3) we have

$$\frac{\partial \varphi}{\partial y} = G\alpha \left(\frac{\partial \psi}{\partial x} - y \right), \qquad -\frac{\partial \varphi}{\partial x} = G\alpha \left(\frac{\partial \psi}{\partial y} + x \right)$$

Eliminating ψ between these two expressions (differentiate the first w.r.t. y, the second w.r.t. x and subtract) we find the stress function to be the solution of

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -2G\alpha \tag{VII.5}$$

φ is a function introduced by Prandtl for solving torsion problems, and so is known as the *Prandtl stress function.* Torsion problems will often be solvable via the inverse method, in a similar manner to that used for plane stress and plane strain. In other words, a Prandtl stress function will be suggested and shown to solve equation (VII.5). Then the boundary conditions for the suggested solution will be evaluated and the problem to which it is the solution discovered. It is interesting to observe that two possible functions, φ and ψ could both be used for this. Although ψ leads to Laplace's equation, which is somewhat easier to solve, it turns out that the boundary conditions are more easilly expressed for Prandtl's function φ, so this is more commonly used. It is easy to see that both ψ and φ are solutions of the biharmonic equation.

VII.3 Boundary conditions

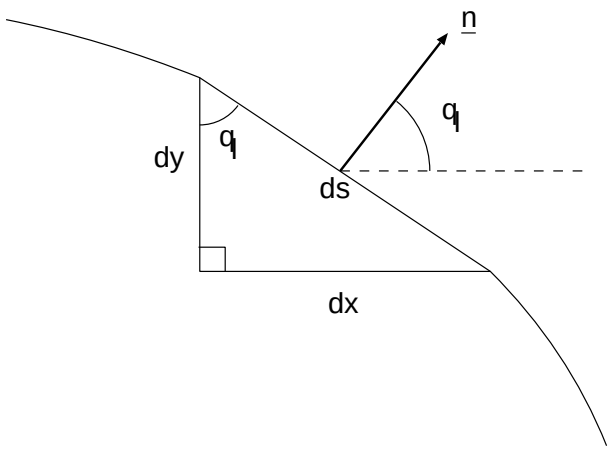

Figure 2: Close-up of a small section of the boundary ds. The angle corresponding to the direction cosine l is shown : $\cos\theta_l = \frac{dy}{ds}$. Similar arguments apply to the direction cosine m.

Let us assume that the bar is being warped by being twisted at the end. This implies that the outer, lateral surface has no applied force, so $\bar{X} = \bar{Y} = \bar{Z} = 0$ in equation (V.13). This surface has $n = 0$ (the surface normal vector is always in the xy plane), and so the first two equations (V.13) are automatically satisfied, leaving

$$\tau_{xz} l + \tau_{yz} m = 0$$

The direction cosines l and m can be written in terms of the gradient of the boundary as

$$l = \frac{dy}{ds}, \qquad m = -\frac{dx}{ds}$$

(see figure 2). Substituting these and the expressions for the stresses, we find

$$\frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial x}\frac{dx}{ds} = \frac{d\varphi}{ds} = 0 \tag{VII.6}$$

Integrating this we have $\varphi = C$ along the boundary, with C an arbitary constant which we can take equal to zero for most cases.

What does this mean? The equation for a circle is

$$x^2 + y^2 = a^2$$

From this, the quantity

$$x^2 + y^2 - a^2$$

will be zero around the edge of a bar of circular cross-section. This implies that the function

$$\varphi = A(x^2 + y^2 - a^2)$$

is a suitable stress function for the torsion of such a bar. In general, if we can write an equation describing the perimeter of the bar, then we can use this as the stress function for the torsion problem. Conversely, given the stress function we can determine the shape of the bar. Hence the function

$$\varphi = B(x - \sqrt{3}y)(x + \sqrt{3}y)(h - x)$$

is a suitable stress function for torsion of a triangular bar.

VII.4 Imposed torsion

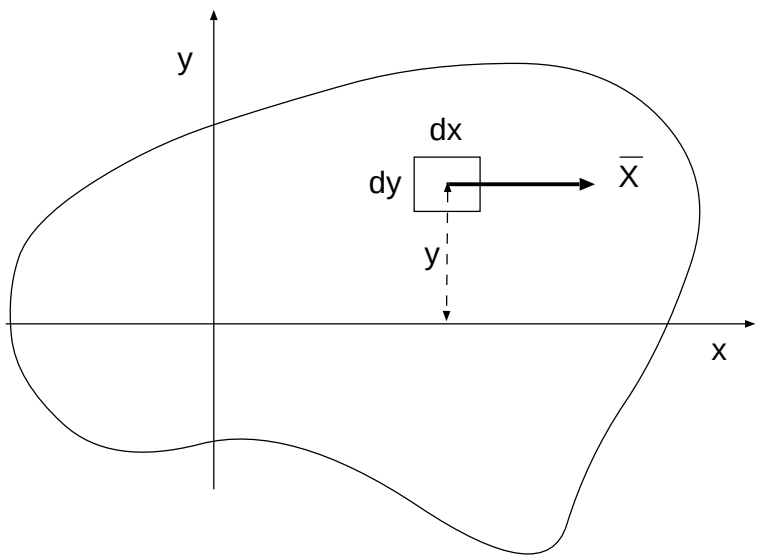

Figure 3: Moment on the end of a bar. The force on a section $dx\,dy$ in the x-direction is $\bar{X}$, so the moment is $y\bar{X}$.

Now, what about the ends? The end surfaces are described by the vectors $(0, 0, \pm 1)$, and so the boundary conditions become

$$\bar{X} = \pm\tau_{xz}, \qquad \bar{Y} = \pm\tau_{yz}$$

using a rh screw convention. Fairly obviously these forces provide a torque which is distorting the bar. The moment arm for $\bar{X}$ is y (see figure 3), that for $\bar{Y}$ is x, and so the couple M is given by

$$M = \iint (\bar{Y}x - \bar{X}y)dxdy = \iint \left(-\frac{\partial\varphi}{\partial x}x - \frac{\partial\varphi}{\partial y}y\right) dxdy$$

where the integration is over the ends of the bar. Integrating the two parts of this this by parts we find

$$M = 2\iint \varphi dxdy \tag{VII.7}$$

This introduces a double integral that must be evaluated. Integration is often introduced as a method of summing under a curve, by dividing the area into small rectangles of width dx and height $f(x)$. A double integral is very similar, but evaluates the function within an area bounded by a curve (or sometimes multiple curves). The method is illustrated in figure 4. The area is divided into strips of width dx, and the function integrated dy across the individual strip from top to bottom. Then the strips themselves are integrated dx.

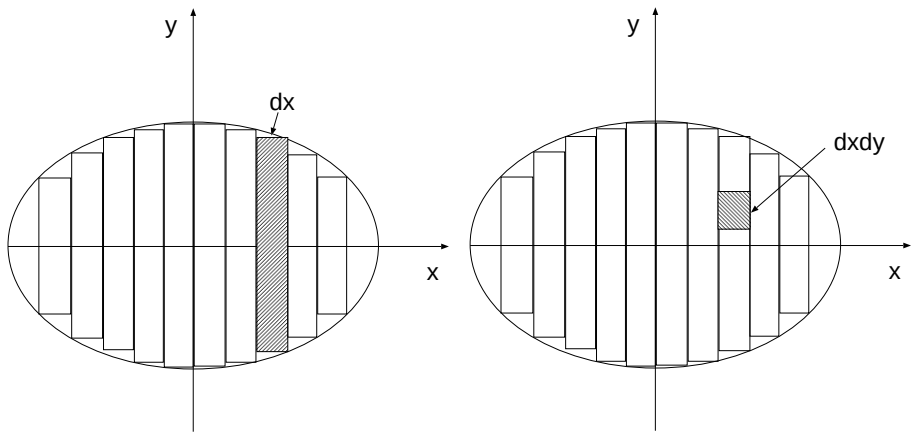

Figure 4: Illustration of a double integral. The area within the curve is split into elements $dx \times dy$, integrated dy then dx

As an illustration of this, let us evaluate

$$I = \iint y^2 dxdy$$

over an area A bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we integrate dy first, exchanging the order, each rectangle goes from

$$y = -b\sqrt{1 - \frac{x^2}{a^2}} \qquad \text{to} \qquad y = +b\sqrt{1 - \frac{x^2}{a^2}}$$

Then we will integrate dx from $x = -a$ to $x = +a$. Thus

$$\begin{aligned} I &= \int_{x=-a}^{x=a} \left[\int_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{y=+b\sqrt{1-\frac{x^2}{a^2}}} y^2 dy \right] dx \\ &= \int_{x=-a}^{x=a} \left[\frac{y^3}{3} \right]_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{y=+b\sqrt{1-\frac{x^2}{a^2}}} dx \\ &= \frac{2b^3}{3} \int_{x=-a}^{x=a} \left(1 - \frac{x^2}{a^2}\right)^{3/2} dx \\ &= \frac{\pi a b^3}{4} \end{aligned}$$

[Q.VII.3] Prove that the resultant of these forces over the ends of the bar is zero.

VII.5 Elliptic Cross section

The boundary of a bar of elliptic cross section can be represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we take a stress function of the form

$$\varphi = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

then this satisfies equation (VII.5) with

$$A = -\frac{a^2 b^2}{a^2 + b^2} G\alpha$$

We can eliminate G (and α) from this by evaluating the moment. From equation (VII.7),

$$M = 2A\left(\iint \frac{x^2}{a^2} dxdy + \iint \frac{y^2}{b^2} dxdy - \iint dxdy\right)$$

The last expression here is just the area of the end, πab. The other two represent the moment of inertia of an ellipse,

$$\iint x^2 dxdy = I_y = \frac{\pi b a^3}{4}, \qquad \iint y^2 dxdy = I_x = \frac{\pi a b^3}{4}$$

thus

$$M = -A\pi ab$$

Substituting this for A gives the stress function

$$\varphi = -\frac{M}{\pi ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

[**Q.VII.4**] Evaluate τ_{xz} and τ_{yz} for an elliptical bar under torsion.
[**Q.VII.5**] Show that the following function

$$\varphi = B(x - \sqrt{3}y)(x + \sqrt{3}y)(h - x)$$

is a valid stress function for an bar under torsion whose cross-section is an equilateral triangle. What is the value of B? Where is the maximum stress, and what is its value?