Linear Elasticity

Reading : Chou & Pagano 8.3 – 8.7, 9.2, 10.1 – 10.3

VI.1 More on Strain.

We have introduced a tensor with 9 components (6 independent) to describe stress. We have also seen that 6 independent components can describe the strain. Can we make these into a tensor? The answer is yes. This tensor takes the form

$$\epsilon = \begin{pmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_z \end{pmatrix} \tag{VI.1}$$

We can of course find principle values and principle directions of strain from this matrix in the same way, not to mention drawing Mohr's circles diagrams.

Can we get there more directly? We will use summation convention. A strain tensor represents the distortion produced in the body by the applied forces. This implies that any point in the undistorted body x_i moves to a new location x_i'. The displacement of this point is of course

$$u_i = x_i' - x_i$$

and we will call u_i the *displacement vector*. Since the vector x_i' is itself a function of x_i, this implies $u_i(x_i)$, and we can write

$$du_i = \frac{\partial u_i}{\partial x_k} dx_k \tag{VI.2}$$

using the chain rule for differentiation.

Now let us consider two neighbouring points x_i^A and x_i^B. Initially their separation vector is dx_i, whilst after the deformation their separation vector is dx_i'. These are related thus :

$$dx_i' = dx_i + du_i$$

So how far apart are they? Initially their separation is $dl = \sqrt{dx_i^2}$, or $dl^2 = dx_i^2$. After the distortion,

$$\begin{aligned} dl'^2 = dx_i'^2 &= (dx_i + du_i)^2 \\ &= dl^2 + 2dx_i du_i + du_i^2 \end{aligned}$$

Using (VI.2) we can eliminate du_i

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l$$

We note two things about this :

1. The second term on the r.h.s. is symmetrical w.r.t exchange of the indices i, k. This implies we can rewrite it as follows

$$2\frac{\partial u_i}{\partial x_k}dx_i dx_k = \frac{\partial u_i}{\partial x_k}dx_i dx_k + \frac{\partial u_k}{\partial x_i}dx_i dx_k$$

2. The third term is similarly symmetrical, so we can swap around the indices

$$\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}dx_i dx_l = \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}dx_i dx_k$$

This being so we can rewrite this equation

$$dl'^2 = dl^2 + 2\epsilon_{ik}dx_i dx_k$$

where

$$\epsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

Finally, if the displacements are small, we can ignore this last term, so

$$\epsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{VI.3}$$

Unfortunately we now have two slightly conflicting notations for components of strain. Earlier we introduced the variables u, v, w for the x, y, z components of displacement, in terms of which the shear strain

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{VI.4}$$

Now we have the vector u_i representing the displacement of point x_i, in terms of which

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{VI.5}$$

However the context should show exactly which u we are using at any one time. In fact the only real confusion arises from the factor of 1/2 in equation (VI.5) versus (VI.4), and this is just a question of where we choose to introduce this factor.

[**Q.VI.1**] Sketch the Mohr's circles for plane strain.
[**Q.VI.2**] Show that the two definitions (VI.1) and (VI.3) are compatible.

VI.2 Stress-Strain Relationships

We will start by considering the simplest possible case. If we take a rod of material and stretch it, how does it behave? This is equivalent to asking what the relationship is between the strain ϵ and the stress σ. If there is a one-to-one relationship between the stress and the strain, then increasing the load on the sample will stretch it further, whilst releasing the load will return the sample to its original state. The material is therefore elastic. For many cases, we can take the relationship to be linear, i.e. $\sigma = K\epsilon$, so the material is referred to as being *linear elastic*. However not all materials behave in this way. Figure 1 also shows the stress-strain relation for an ideal plastic material. Here there is no longer a 1-to-1 relation between the stress and the strain, in fact an applied stress σ_Y will cause an arbitary strain to occur. Loads up to σ_Y will have no effect, when the load reaches σ_Y the sample will begin to deform, moving from left to right along the diagram. However if the load is removed the sample will not return to its original state.

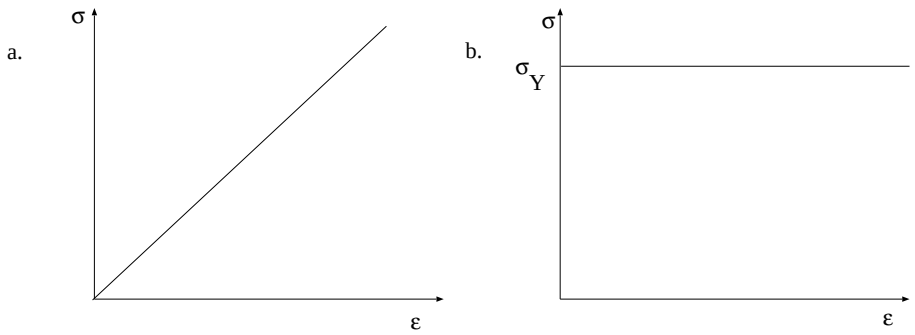

Figure 1: Stress-strain curves. a. represents a linear elastic material. A sample being loaded and unloaded will run back and forth along this straight line. b. represents a plastic material.

Real materials tend to have a more complex behaviour than this. Typically a stress-strain relationship for a sample might look like figure 2. The material behaves elastically up to some limiting stress, beyond which it deforms plastically. In figure 2 the sample behaves linearly to a limiting stress σ_Y, and then behaves plastically beyond there. Some materials may exhibit non-linear elastic behaviour though, with a yield stress at σ'_Y.

In 3d, things become rather more complex. The *yield condition* for a material is a relationship between the stress components that must be satisfied for the onset of plastic behaviour at a point. This can be expressed as the equation

$$f(\sigma_{ij}) = C_Y$$

where C_Y is the yield constant. If the material is isotropic then the yield condition has to be independent of direction. This can be accomplished by expressing the yield condition in terms of the principle stresses,

$$f(\sigma_I, \sigma_{II}, \sigma_{III}) = C_Y$$

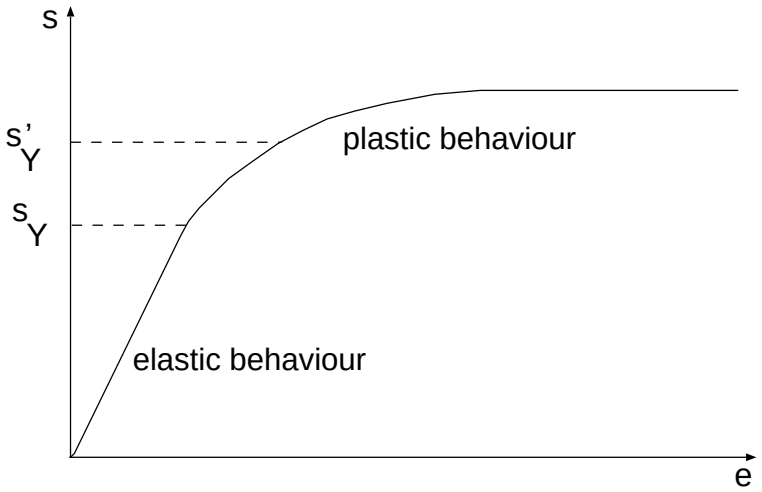

Figure 2: Typical stress-strain diagram for a material. The yield stress σ_Y divides the linear, elastic region from the more complex plastic region.

where we have arranged these so that $\sigma_I > \sigma_{II} > \sigma_{III}$.

The *Tresca yield condition* asserts that yielding occurs when the maximum shear stress reaches the value C_Y. This can be expressed as

$$\frac{1}{2}(\sigma_I - \sigma_{III}) = C_Y$$

[**Q.VI.3**] Sketch Mohr's circles for a state of simple tension. Use this to show that the Tresca yield condition can be expressed as

$$\sigma_I - \sigma_{III} = \sigma_Y$$

where σ_Y is the yield stress in simple tension.
[**Q.VI.4**] Eigenvalues of σ_{ij} are calculated from the characteristic equation $\det(\sigma_{ij} - \delta_{ij}\sigma) = 0$. Show that this can be written in the form

$$\sigma^3 - I_\sigma \sigma^2 + II_\sigma \sigma - III_\sigma = 0$$

I_σ, II_σ and III_σ are the *first, second* and *third stress invariants.* What are their values?

VI.3 Hooke's Law

The previous section kicked off with a discussion of the relationship between stress and strain for a 1d case. In the general 3d case, the relationship between stress and strain is

referred to as the *constitutive equation.* For a linear elastic solid the constitutive relation is *(the generalised) Hooke's Law.*

Generalised? Both σ and ϵ are second rank tensors, so the most general relationship between them has to look something like this :

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{VI.6}$$

Since each index of C can take one of three values, there are therefore 81 components to C_{ijkl}. However we know that ϵ_{kl} is a symmetric tensor, so we must be able to write

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = C_{ijlk}\epsilon_{lk} \qquad \text{and} \qquad C_{ijkl} = C_{ijlk}$$

which reduces the number of independent components to 54. Also, σ_{ij} is also symmetric, so

$$C_{ijkl} = C_{jikl}$$

This reduces the number of components to 36, which is still rather a lot, but is a bit more manageable. This is the most general form : if the material in question has particular types of symmetries, then many more of these components can be shown to be either zero or to be the same. The most complete symmetry possible is for the material to be isotropic. Under these circumstances, there are in fact only 2 independent components of C_{ijkl}, and Hooke's law (VI.6 can be written

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{VI.7}$$

where μ is the shear modulus G

$$\mu = G = \frac{E}{2(1+\nu)}$$

and

$$\lambda = \frac{E}{1+\nu}\frac{\nu}{1-2\nu}$$

so

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\sigma_{kk}\right) \tag{VI.8}$$

[Q.VI.5] Show that in the general case C_{ijkl} has 36 independent components.
[Q.VI.6] Show that equation (VI.8) can be inverted to give

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

C_{ijkl} is in fact a 4th rank tensor. This means it transforms according to

$$C_{mnop} = \Lambda_{mi}\Lambda_{nj}\Lambda_{ok}\Lambda_{pl}C'_{ijkl}$$

If the material properties are to be isotropic then C_{ijkl} has to be the same under any transformation, i.e.

$$C_{ijkl} = C'_{ijkl}$$

so that

$$\sigma'_{ij} = C_{ijkl}\epsilon'_{kl}$$

The proof of (VI.7) for an isotropic material is straightforward, although long. We will outline part of it here. We start off by demonstrating that for an isotropic material the principal axes of the stress and the strain tensors are aligned. Let us assume that our coordinate system aligns with the principal axes of the strain tensor, that is

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}$$

Equation (VI.6) shows that the off-diagonal components of σ, say

$$\sigma_{yz} = C_{yzxx}\epsilon_{xx} + C_{yzyy}\epsilon_{yy} + C_{yzzz}\epsilon_{zz} \tag{VI.9}$$

Now let us apply a rotation of 180° about the z-axis. This is given by the transformation matrix

$$\Lambda_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Applying this to σ_{yz} gives

$$\sigma'_{yz} = \Lambda_{ky}\Lambda_{lz}\sigma_{kl} = -\sigma_{yz}$$

However the components of ϵ do not change under this transformation. So in the new coordinate system, equation (VI.6) becomes

$$\sigma'_{yz} = -\sigma_{yz} = C_{yzxx}\epsilon_{xx} + C_{yzyy}\epsilon_{yy} + C_{yzzz}\epsilon_{zz} \tag{VI.10}$$

Comparing this with equation (VI.9) we see that this implies $\sigma_{yz} = -\sigma_{yz}$, i.e. $\sigma_{yz} = 0$. The other components follow in a similar manner, so we see that in this coordinate system (given by the principal axes for the strain tensor) the stress tensor is itself diagonal. In other words, the stress and strain tensors are aligned.

One possible proof that there are only 2 independent coefficients of C_{ijkl} for an isotropic material would now consider the transformational properties of C_{ijkl} under various rotations. We will take a more intuitive route and ask, what functions of ϵ could we combine to get a σ which has the same principal axes as ϵ? Working in the principal axes coordinate system we want to find functions of ϵ that do not generate off-diagonal components in σ, i.e.

$$\begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} = f\left[\begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}\right]$$

Clearly, any scalar multiple of ϵ, $\mu\epsilon_{ij}$ will work. So will any scalar multiplied by the identity matrix, so $\lambda\delta_{ij}\epsilon_{kk}$ is another possibility. However this exhausts the possibilities. (Actually, higher powers of ϵ would work as well, but we are considering *linear* elasticity here). Thus the constitutive relation for an isotropic material in the linear elastic regime is indeed

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij}$$

[**Q.VI.7**] An electric-resistance strain gauge measures elongational strain in one direction on the surface of a sample. If the gauge is placed at an angle θ to the x-axis, show that the measured strain ϵ_G is given by

$$\epsilon_G = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$

If three strain gauges are placed, along the x-axis, and at $\pm 45°$ to the x-axis, derive expressions relating the measured stresses ϵ_A, ϵ_B, ϵ_C to the stress components at that point on the sample.