2d solutions – Cartesian Coodinates

Reading : Chou & Pagano 4.1 – 4.3, 5.1 – 5.3
Timoshenko & Gere 2.8, 2.9, 2.14 – 2.16

III.1 Plane Stress and Plane Strain

We will start by looking at 2d solutions to problems of elasticity. There are 2 limiting cases where the solution will be 2d. The first corresponds to that of an infinitely thin plate, which we can take to lie in the xy plane, with the applied forces also being in this plane. This implies that the only components of stress we are interested in are $\sigma_x(x,y), \sigma_y(x,y), \tau_{xy}(x,y)$. This is referred to as *plane stress*.

At the opposite extreme we can consider loadings on an infinitely long prism – for instance for a tube under pressure. If the prism is along the z axis, then any displacements are in the xy plane, so

$$u = u(x,y), \qquad v = v(x,y), \qquad w = 0$$

From this

$$\begin{aligned} \gamma_{yz} &= \frac{\delta v}{\delta z} + \frac{\delta w}{\delta y} = 0 \\ \gamma_{xz} &= \frac{\delta u}{\delta z} + \frac{\delta w}{\delta x} = 0 \\ \epsilon_z &= \frac{\delta w}{\delta z} = 0 \end{aligned} \tag{III.1}$$

From Hooke's Law we can show

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right) \tag{III.2}$$

and also

$$\tau_{xz} = 0, \qquad \tau_{yz} = 0$$

Once again this problem is fully defined once we have found $\sigma_x(x,y), \sigma_y(x,y), \tau_{xy}(x,y)$. This state is referred to as *plane strain*. Of course, an infinitely long body is an abstraction, in the real world we are talking about reasonably long bodies with ends. If an appropriate stress σ_z is applied across the ends however this is a reasonable approximation.

These two cases therefore represent the extremes, when the stresses are acting in 2d (plane stress) and when the strains are occurring in 2d (plane strain). To start with we will concentrate on the case of plane stress.

[**Q.III.1**] Derive equations (III.2), (III.2).
[**Q.III.2**] Define plane strain, and give some examples.

III.2 Equations of Elasticity in 2d

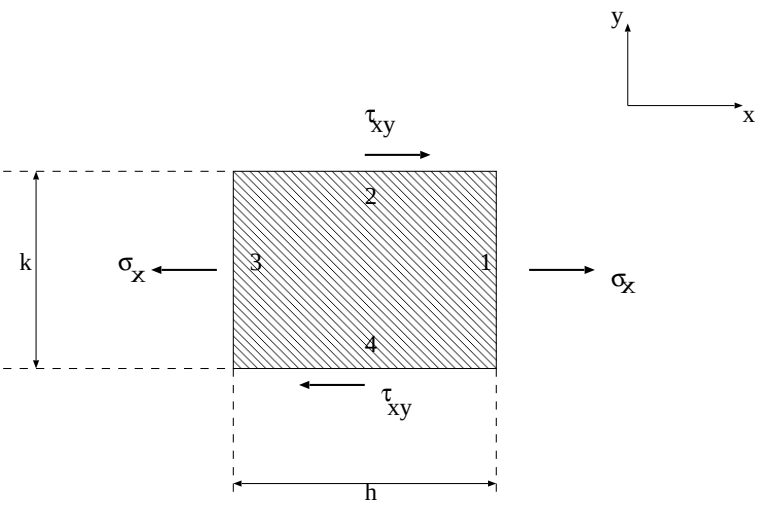

Figure 1: x component of forces acting on a small element of the domain, in 2d.

We can derive the equations of equilibrium in 2d in a simple manner, by applying a force balance to a small rectangular element (figure 1). In the x direction normal (σ_x) and shear (τ_{xy}) stresses must balance. In 2d the face areas correspond to the sides of the rectangle, so in the x direction we have

$$(\sigma_x)_1 k - (\sigma_x)_3 k + (\tau_{xy})_2 h - (\tau_{xy})_4 h = 0$$

for a case where there are no body forces applied. We can rewrite this as

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} = 0$$

In the limit as $h, k \to 0$ we get the partial differential equation

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} &+ \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_y}{\partial y} &+ \frac{\partial \tau_{xy}}{\partial x} = 0 \end{aligned} \tag{III.3}$$

with the second equation coming from a force balance along the y direction.

This gives 2 equations for 3 unknown quantities, so we need an extra equation from somewhere. This comes from considering the deformation of the body, i.e. the strains. In 2d problems we have 3 strain components :

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Differentiating these equations as follows, the first twice with respect to y, the second twice w.r.t. x and the third once w.r.t. x, once w.r.t. y, and then combining them gives the equation

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{III.4}$$

This equation is called the *condition of compatibility.*

This gives us another equation to solve, but in order to do so we need to rewrite this in terms of the stresses. For plane stress we can use equations (I.9) to relate stresses and strains as follows :

$$\begin{aligned} \epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad & \epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} = \frac{1}{G}\tau_{xy} \end{aligned} \tag{III.5}$$

Substituting in equation (III.4) we find

$$\frac{\partial^2}{\partial y^2}\left(\sigma_x - \nu\sigma_y\right) + \frac{\partial^2}{\partial x^2}\left(\sigma_y - \nu\sigma_x\right) = 2\left(1+\nu\right)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{III.6}$$

Finally we can eliminate the term in τ_{xy} on the rhs. If we differentiate the first equation (III.3) wrt. x, and the second wrt y, and add them, then we get the equation

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

as an expression for $\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$. Substituting back in (III.6), we finally find

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = 0 \tag{III.7}$$

III.3 Boundary Conditions

We now have a set of 3 equations to solve for 3 unknowns :

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) &= 0 \end{aligned} \tag{III.8}$$

The precise solution to these equations will depend on the imposed forces on the object which are doing the distorting, i.e. on the boundary conditions. Specifically the stress components at the surface of the body must be in equilibrium with the imposed external

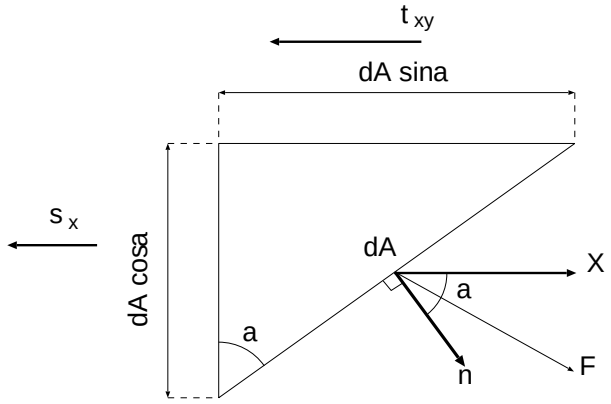

Figure 2: Force balance in x direction at boundary.

forces. We can deal with this balance in the same way as we did for the internal balance, by considering the forces exerted on the small triangle shown in figure 2. We are looking at a small area of the boundary dA, which makes up the hypotenuse of the triangle. Simple trigonometry gives $dA\cos\alpha$ and $dA\sin\alpha$ as the areas of the other two sides, with α being the angle between the normal to dA and the x direction. Thus a force balance in the x direction shows that

$$dA\sigma_x\cos\alpha + dA\tau_{xy}\sin\alpha = F_x$$

with F_x being the imposed force component in this direction. Dividing through by dA and writing $\overline{X}$ for the surface force per unit area in this direction, we have

$$\overline{X} = \sigma_x\cos\alpha + \tau_{xy}\sin\alpha \tag{III.9}$$

and of course a similar equation for the y direction

$$\overline{Y} = \sigma_y\sin\alpha + \tau_{xy}\cos\alpha \tag{III.10}$$

III.4 Stress function

Solving 3 pde's simultaneously is very demanding. It turns out to be easier if we introduce a new function ϕ called the *stress function* (sometimes the *Airy stress function*). For the simple case that we are considering here, where there are no body forces, we can define ϕ so that

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \tag{III.11}$$

If we substitute these relations into equation (III.7) then we end with the equation

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \tag{III.12}$$

Don't worry, this is not as bad as it looks. Another way of writing this is to introduce a new operator, the *laplacian*

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in which case it can be written as

$$\nabla^2\nabla^2\phi = 0$$

[Q.III.3] Prove that ϕ as defined by equations (III.11) satisfies the equations of equilibrium (III.3).

We can (and will) solve this equation in a number of ways. The easiest is to observe that a polynomial function of 2nd degree, i.e. of the form

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2$$

is a solution (just substitute it in the equation). Such a solution turns out to be very useful for cases involving long rectangular strips. We can easilly show that the stress components are

$$\sigma_x = c_2, \qquad \sigma_y = a_2, \qquad \tau_{xy} = -b_2$$

so this is a solution where the stresses are uniform. Of course the function ϕ also has to satisfy the boundary conditions, so this would be appropriate for a case where a rectangular body is experiencing a uniform traction force or uniform applied shear. Often however we will want to work in the opposite direction. For instance, if our rectangular strip is experiencing a uniform compressive force of 5N/m^2 in the x direction at one end and the same in the $-x$ direction at the other end, then we know

$$\sigma_x = c_2 = 5\text{N/m}^2, \qquad \sigma_y = \tau_{xy} = 0$$

and so our stress function is

$$\phi = 2.5y^2$$

[Q.III.4] Show that the polynomial

$$\phi_3 = \frac{a_3}{3!}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{3!}y^3$$

is a stress function. If all the constants except d_3 were set to zero in this, what would the stress pattern correspond to?

III.5 Inverse method

Solution of the equations of elasticity for given boundary conditions is frequently very difficult to do in a straightforward manner, i.e. by solving the biharmonic equation and then introducing the boundary conditions. Frequently we will use a technique known as the *inverse method*, in which we will start by guessing a solution, show that the guess is a solution to the biharmonic equation, and then find out what boundary conditions can be satisfied by this solution.

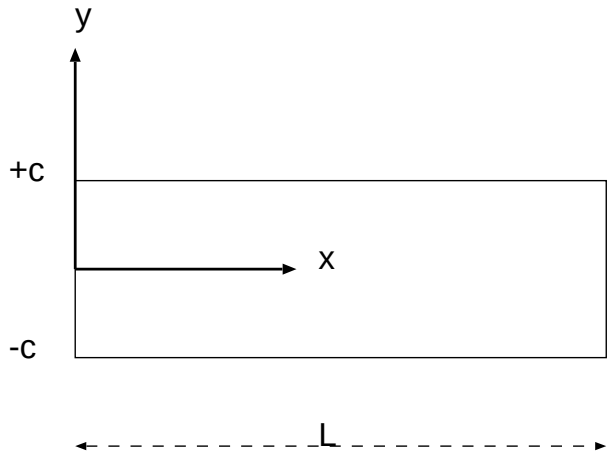

Figure 3: Domain for the stress function in section III.5

As an example, consider the function

$$\phi = \frac{3F}{4c}\left[xy - \frac{xy^3}{3c^2}\right] - \frac{P}{4c}y^2$$

Is this a valid stress function? We can evaluate the biharmonic equation in two stages. Firstly,

$$\begin{aligned} F &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \\ &= -\frac{3F}{2c^3}xy - \frac{P}{2c} \end{aligned}$$

and so

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

proving that $\nabla^2\nabla^2\phi = 0$ for this function, so it is a valid stress function.

Now we must determine the problem for which this is a solution. Let us work out the stresses on the domain shown in figure 3. From the stress function

$$\begin{aligned}
\sigma_x &= \frac{\partial^2\phi}{\partial y^2} = -\frac{3F}{2c^3}xy - \frac{P}{2c} \\
\sigma_y &= \frac{\partial^2\phi}{\partial x^2} = 0 \\
\tau_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{3F}{4c}\left[1 - \frac{y^2}{c^2}\right]
\end{aligned}$$

Evaluating these along $y = \pm c$, $\tau_{xy} = 0$, so these edges could represent the edge of a beam with no applied load. Along $x = 0$,

$$\sigma_x = -\frac{P}{2c}, \qquad \tau_{xy} = -\frac{3F}{4c}\left[1 - \frac{y^2}{c^2}\right]$$

which is the combination of a force to the right P (along $+x$) and F down (along $-y$). In fact this potential is appropriate for the solution of a cantilever loaded with a compressive force P and a shearing load F, as shown in figure 4.

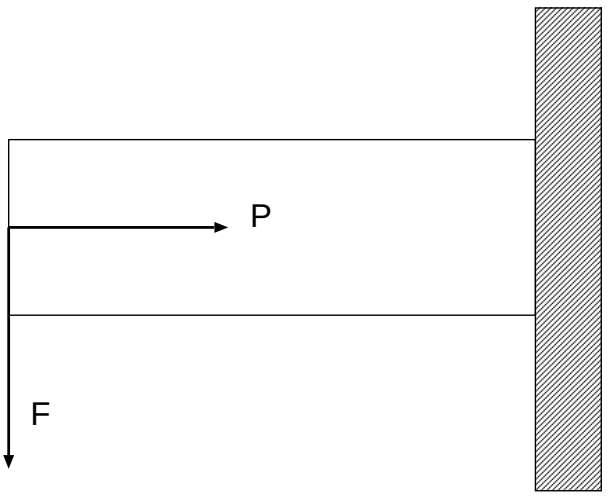

Figure 4: Domain for the stress function in section III.5

We note the following points :

1. The applied load could be considered separately as a compression and a shear force. Similarly the solution is the *superposition* of two functions that are themselves solutions of the problem. This is an example of the *principle of superposition.*

2. The shear force has a parabolic distribution over the l.h. end. However by *St Venant's principle*, the exact distribution of the load is unimportant away from the point of application. Hence the solution away from this area (i.e. distance c away from the end) will be the same no matter what the shear distribution actually is.