Factorisation (Ref: Croft & Davison Ch.11)

Factors

Factors of a number are smaller numbers that multiply together to give the original number

Eg 12 – factors are (12 x 1) or (6 x 2) or (3 x 4)

18 – factors are (18 x 1) or (9 x 2) or (6 x 3)

Algebraic factors

Algebraic factors are terms that multiply together to give the original expression

Eg $4y^3$ – factors are $(4 \times y \times y \times y)$ or $(4 \times y \times y^2)$ or $(2 \times 2 \times y \times y \times y)$
$15a^2b^3$ – factors are $(5 \times 3 \times a^2 \times b^3)$ or $(5 \times 3 \times a \times a \times b \times b \times b)$

Prime factors

Both numeric and algebraic expressions can have a set of prime factors. Prime factors are those where each number or term in the set of factors cannot be further factorised.

Eg 12 – in the set of factors (3 x 4), the 3 is a prime number, but the 4 can be further factorised to give 2 x 2, so the prime factors would be (2 x 2 x 3)

$4y^3$ – prime factors are $(2 \times 2 \times y \times y \times y)$

Common factors

Common factors are those that appear in two or more numbers or terms

Eg 12 – factors are $2 \times 2 \times 3$
15 – factors are 5×3
} common factor is 3

$4a^2b^2$ – factors are $2 \times 2 \times a \times a \times b \times b$
$2a^3$ – factors are $2 \times a \times a \times a$
} common factors are $2 \times a \times a$ or $2a^2$

Given $3x(x^2 + 2x)$ it is possible to expand the bracketed terms to give separate terms

Eg $3x(x^2 + 2x) = 3x^3 + 6x^2$

It follows therefore that given the separate terms, it must be possible to work back to the bracketed version – this is called factorisation.
Whenever you are instructed to factorise an expression, the first step is to look for any common factor and take it out. **nb** – take out a common factor means divide through by the common factor **NOT** subtract it!

Eg $3y^3 + 6y = 3 \times y \times y \times y + 3 \times 2 \times y$ so $3y$ is common to both terms

The common factors go outside the bracket, what is left when each term is divided by the common factor goes inside the bracket

Eg $3y$ is common factor so $\frac{3 \times y \times y \times y}{3 \times y} + \frac{3 \times 2 \times y}{3 \times y} = 3y(y^2 + 2)$

Worked example 1

Factorise the following expression

$15y^3 + 5y^2 - 10y = 5 \times 3 \times y \times y \times y + 5 \times y \times y - 5 \times 2 \times y$

so $5y$ is common to all terms

when factorised $= \frac{5 \times 3 \times y \times y \times y}{5y} + \frac{5 \times y \times y}{5y} - \frac{5 \times 2 \times y}{5y}$

$= 5y(3y^2 + y - 2)$

Worked example 2

Factorise the following expression

$9a^3 b^3 + 6a^2b - 3ab = 3 \times 3 \times a \times a \times a \times b \times b \times b + 3 \times 2 \times a \times a \times b - 3 \times a \times b$

so $3ab$ is common to all terms

when factorised $= \frac{3 \times 3 \times a \times a \times a \times b \times b \times b}{3ab} + \frac{3 \times 2 \times a \times a \times b}{3ab} - \frac{3 \times a \times b}{3ab}$

$= 3ab(3a^2b^2 + 2a - 1)$

Note that when the common factor is the whole of a term, on factorisation this gives the figure 1 not zero! (as in third term in example given above)

It does not matter how complicated the expression looks, common factors can always be identified if the terms are expressed as prime factors first.

Once expressed as prime factors, compare each term to the first term, if a factor is not in the first term it cannot possibly be common!

Worked example 3

Factorise the following expression

$12x^3y^2z^3 - 18x^3y^2z + 24xy^3z^2 = 6xy^2z\,(2x^2z^2 - 3x^2 + 4y)$

Factorisation – Trinomials

If the expression $(x + 3)(x + 2)$ were simplified by multiplying the brackets the result would give $x^2 + 3x + 2x + 6$ which, when tidied up, would give $x^2 + 5x + 6$. It follows that given the expression $x^2 + 5x + 6$, it must be possible to reverse the process to obtain the bracketed form. This is known as factorising a trinomial. The fact that two like terms have been combined to give the middle term of the trinomial makes factorising it a little more complicated than other methods of factorisation.

For any expression, once any common factor has been identified and taken out, if there are three terms remaining, they may be factorised providing they form a pattern known as a 'trinomial'. A trinomial (in one unknown only) must have an x^2 term, an x term and a numerical term – in that order, for example $x^2 + 5x + 6$. If the three terms do form a trinomial they *may* be factorised as follows (not all trinomials are factorisable!).

Draw two brackets side by side ( ) ( )
Factors of the first term go into the first position in both brackets
Factors of the last term go into the second position in both brackets
Appropriate signs are added between the terms in both brackets

Worked example 1

Factorise the expression $x^2 + 5x + 6$

There is no common factor
There are three terms in the expression
The three terms are of trinomial form

Factors of first term are $x \times x$
Factors of last term are 3×2

$\therefore\ x^2 + 5x + 6 = (x + 3)(x + 2)$

The sign placed between the terms in the brackets must be such that the correct middle term is obtained if the brackets are multiplied back - this must always be done as a check! Looking at the signs between the terms of the trinomial can *sometimes* help with identifying the appropriate signs for the brackets as follows:

Signs in trinomial	Signs in brackets	Example
+ +	(+) (+)	$x^2 + 6x + 8 = (x + 2)(x + 4)$
- +	(-) (-)	$x^2 - 5x + 6 = (x - 2)(x - 3)$
± -	one + and one -	$x^2 - 3x - 10 = (x + 2)(x - 5)$
		$x^2 + 2x - 3 = (x + 3)(x - 1)$

The coefficient of the x^2 term in all examples so far has been 1, this simplifies the factorisation. However, both the coefficient of the x^2 term and the numerical term may have more than one set of factors. It is then essential that the correct combination of factors is selected. Often this will be by 'trial and error' however, when there are more than one pair of factors for a number, try the pair with the smallest numerical difference first. This won't always be the correct choice but it does work more often than not!! These factors are called the 'middle of the road' factors

Worked example 2

Factorise the expression $12x^2 + 11x - 15$

The expression has no common factor
The expression has three terms
The three terms do form a trinomial

Factors of coefficient of x^2 term	12×1 difference $12 - 1 = 11$
	6×2 difference $6 - 2 = 4$
	4×3 difference $4 - 3 = 1$
Factors of numerical term	15×1 difference $15 - 1 = 14$
	5×3 difference $5 - 3 = 2$

So pairs of factors to try first are 4×3 for x^2 term and 5×3 for numerical term

$$12x^2 + 11x - 15 = (4x + 3)(3x - 5)$$

Always multiply the brackets back to ensure the correct middle term has been achieved. In this example multiplication of brackets gives

$$12x^2 + 9x - 20x - 15$$

when tidied up gives $12x^2 - 11x - 15$

but notice that the middle term is $-11x$ whereas in the original expression it was $+11x$. This shows that the correct pairs of factors have been chosen, but the signs in the brackets are wrong. When they are changed the correct solution is achieved

$$\therefore\ 12x^2 + 11x - 15 = (4x - 3)(3x + 5)$$

If the trinomial is in two unknowns it must have an x^2 term, an xy term and a y^2 term. It can then be factorised in the same way as for trinomials in one unknown.

Worked example 3

Factorise the expression $x^2 - 2xy - 8y^2$

Expression has no common factor
Expression has three terms
The three terms do form a trinomial in two unknowns

$\therefore\ x^2 - 2xy - 8y^2 = (x - 4y)(x + 2y)$

Factorisation - Difference of two squares

Once any common factor has been identified and 'taken out', other forms of factorisation may be carried out provided the expression meets certain conditions.

If the expression to be factorised has two separate terms, further factorisation can take place only if the terms form a pattern known as a difference of two squares. This requires each of the two terms to have an exact square root and there MUST be a minus sign between them.

Eg $9 - x^2 = 3 \times 3 - x \times x$

So square root of first term is 3, square root of second term is x and there is a minus sign between the two terms. This expression is therefore a difference of two squares.

$9 + x^2$ is not a difference of two squares as there is a plus sign between the two terms

$12 - a^2$ is not a difference of two squares as 12 does not have an exact square root

$16 - x^3$ is not a difference of two squares as x^3 does not have an exact square root

If the expression does meet the conditions for a difference of two squares, it is factorised as follows:

Draw two brackets side by side ( ) ( )
Determine square root of each term
Place square root of first term in first position of each bracket
Place square root of second term in second position of each bracket
Add signs between the terms in each bracket. One must be + and the other must be – (it doesn't matter which sign goes in which bracket)

Worked example 1

Factorise the expression $9x^2 - 16y^2$

There is no common factor
There are two terms
Each term has an exact square root
There is a minus sign between the two terms so $9x^2 - 16y^2$ is a difference of two squares

$\sqrt{9x^2} = 3x$ and $\sqrt{16y^2} = 4y$

$\therefore\ 9x^2 - 16y^2 = (3x + 4y)(3x - 4y)$

You can check this is correct by multiplying out the brackets again

Worked example 2

Factorise the expression $25a^2 - 36b^2$

There is no common factor
There are two terms
Each term has an exact square root
There is a minus sign between the two terms so $25a^2 - 36b^2$ is a difference of two squares

$\sqrt{25a^2} = 5a$ and $\sqrt{36b^2} = 6b$

$\therefore\ 25a^2 - 36b^2 = (5a + 6b)(5a - 6b)$

Remember that if the two terms have a + sign between them, they form a prime expression and cannot be factorised in any way.

Remember also, that if instructed to factorise an expression, any common factor must be identified and taken out before attempting any other form of factorisation

Worked example 3

Factorise the expression $12x^2 - 75y^2$

Take out common factor of 3

$12x^2 - 75y^2 = 3(4x^2 - 25y^2)$ - contents of bracket form difference of two squares

$\therefore\ 12x^2 - 75y^2 = 3(2x + 5y)(2x - 5y)$ - expression now fully factorised

Once an expression has been factorised, the new expression should be examined to see if any further factorisation can take place.

Worked example 4

Factorise the expression $16x^4 - 81y^4$

$16x^4 - 81y^4 = (4x^2 + 9y^2)(4x^2 - 9y^2)$

However, the contents of the second bracket still form a difference of two squares

$\therefore\ 16x^4 - 81y^4 = (4x^2 + 9y^2)(2x + 3y)(2x - 3y)$

Contents of the first bracket do not still form a difference of two squares because there is a plus + sign between the two terms.

Factorisation - Worksheet 1

Take out the common factors from each of the following:

1. $qt + rt$
2. $rt - 5rs$
3. $3xy + 4xz$

Using difference of two squares factorise the following:

4. $p^2 - q^2$
5. $t^2 - 1$
6. $4 - n^2$

Factorise the following:

7. $x^2 + 3x + 2$
8. $x^2 + 4x + 4$
9. $x^2 - 11x + 24$

Factorisation - Worksheet 2

Take out the common factors from each of the following:

1. $4a + 6ab - 8ab^2$
2. $5h^2 + 10gh - 20g^2h$
3. $2c^2d - 4cd^2$

Using difference of two squares factorise the following:

4. $7 - 7x^2$
5. $2y^2 - 8x^2$
6. $108 - 3x^2$

Factorise the following:

7. $5x^2 + 9x + 4$
8. $6x^2 + 35x - 6$
9. $8x^2 - 19x + 6$